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Dipole Ghosts and Unitarity.

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Summary. — An S-matrix defined by Feynman diagrams in a state space with indefinite norm is considered. The propagators may have, in addition to simple poles, also double poles in momentum space. A unitarization procedure is derived which projects the S-matrix on the physical subspace.

1. – Introduction.

An S-matrix may be given by a set of Feynman diagrams. The propagator can have, in addition to simple poles, also double poles. These double poles are connected with ghost and dipole ghost states of norm zero. As a consequence of this indefinite norm in the state space the S-matrix is not unitary in the usual sense, although pseudounitarity $S^\dagger S = SS^\dagger = 1$ is supposed. There is no physical interpretation of the S-matrix elements, since negative « probabilities » can appear. We construct a positive-definite physical subspace $\overline{\mathscr{H}}$ of the given indefinite state space $\mathscr{H}$. $\overline{\mathscr{H}}$ consists of all states containing no dipole ghosts both before and after scattering (*) [1,2]. The S-operator projected on this subspace $\bar{S} = PSP$ is unitary in the usual sense: $\bar{S}^\dagger \bar{S} = \bar{S}\bar{S}^\dagger = P$. This unitarization method is unique, produces no new divergences as well as no nonanalytic points in the physical range (excepting of course the physical thresholds) (**).

(*) This definition is exact only for simple cases (Sect. **3**, **4**, **5**, **7** and ref. [2]), for the general case see Sect. **6**, **8**.

[1] W. Heisenberg: *Introduction to the Unified Field Theory of Elementary Particles* (London, 1967).

[2] M. Karowski: *Zeits. Naturforsch.*, **24** a, 510 (1967) and MPI preprint (1972).

(**) This unitarization prescription is more general than that of Ascoli and Minardi [3] which only works for the very special case that the eigenstates of the Hamil-

States containing only one ghost are considered in Sect. **3** and **4** for elastic scattering. Section **5** contains some remarks on bound states. In (6) the more complicated case is treated where the states contain two ghosts.

2. – The propagator.

Let us consider a real scalar field $\varphi(x)$ and the propagator

$$\langle 0|T\varphi(x)\varphi(y)|0\rangle = \frac{i}{(2\pi)^4}\int \mathrm{d}^4p \exp[-ip(x-y)]\Delta(p) \tag{2.1}$$

(Feynman integration is always used at the singularities of $\Delta(p)$). As is well known, a particle $n(\boldsymbol{p})$ with mass m is associated with a pole of $\Delta(p)$ at $p^2 = m^2$:

$$\Delta(p) = \frac{R_m(p^2)}{p^2 - m^2}\,. \tag{2.2}$$

This can be seen by the following consideration:
For $x_0 > y_0$ we have

$$\begin{aligned} \langle 0|\varphi(x)\varphi(y)|0\rangle &= \frac{i}{(2\pi)^4}\int \mathrm{d}^4p \exp[-ip(x-y)]\frac{R_m(p^2)}{p^2-m^2} = \\ &= \frac{i}{(2\pi)^4}\int \mathrm{d}^3p \oint_\omega \mathrm{d}p_0 \exp[-ip(x-y)]\frac{R_m(p^2)}{(p_0-\omega)(p_0+\omega)} + \\ &\quad + \text{contribution from other singularities} = \\ &= \int \frac{\mathrm{d}^3p}{2\omega}\langle 0|\varphi(x)|n(\boldsymbol{p})\rangle\langle n(\boldsymbol{p})|\varphi(y)|0\rangle + \dots, \end{aligned} \tag{2.3}$$

with the wave function for the particle $n(\boldsymbol{p})$

$$\langle 0|\varphi(x)|n(\boldsymbol{p})\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}}\exp[-ipx]\,, \qquad p_0 = \omega = \sqrt{m^2 + \boldsymbol{p}^2}\,, \tag{2.4}$$

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tonian belonging to real positive eigenvalues form a positive semi-definite subspace. This is in general not true (4,5).

(3) R. Ascoli and E. Minardi: *Nuovo Cimento*, **8**, 951 (1958); *Nucl. Phys.*, **9**, 242 (1958).

(4) N. Nakanishi: *Phys. Rev. D*, **3**, 1343 (1971).

(5) K. L. Nagy: preprint ITP-Budapest Report No. 302 (1971).

(6) Detailed version of this paper: MPI preprint January 1973, Munich.

if by a suitable normalization $R_m(m^2) = 1$. From the weak asymptotic condition

$$|n(f), \psi\rangle^{\pm} = \underset{x_0 \to \mp\infty}{\text{w-lim}} \; i\int d^3x\, \varphi(x) \overleftrightarrow{\partial_0} f(x)|\psi\rangle^{\pm} \tag{2.5}$$

we get the reduction formula for the S-matrix

$$\begin{aligned} {}^-\langle\psi', n(f)|\psi\rangle^+ &= i\int d^4x\, f^*(x)(\Box + m^2)\, {}^-\langle\psi'|\varphi(x)|\psi\rangle^+ = \\ &= -i\int d^4p\, \tilde{f}^*(p)\, \frac{1}{\Delta(p)}\, \frac{1}{(2\pi)^2}\int d^4x \exp[ipx]\, {}^-\langle\psi'|\varphi(x)|\psi\rangle^+ \end{aligned} \tag{2.6}$$

with $\tilde{f}(p) \sim \delta(p^2 - m^2)$, if $|\psi\rangle^+$ contains no particle $n(f)$.

To a double pole of $\Delta(p)$ at $p^2 = \lambda^2$

$$\Delta(p) = \frac{R_\lambda(p^2)}{(p^2 - \lambda^2)^2} \tag{2.7}$$

there belong two states: a « good » ghost $g(\boldsymbol{p})$ and a « bad » or dipole ghost $d(\boldsymbol{p})$ with

$$\begin{cases} \langle g|g\rangle = \langle d|d\rangle = 0\,, \\ \langle g(\boldsymbol{p}')|d(\boldsymbol{p})\rangle = 2\varepsilon\delta(\boldsymbol{p}' - \boldsymbol{p})\,, \\ \varepsilon = \sqrt{\lambda^2 + \boldsymbol{p}^2}\,. \end{cases} \tag{2.8}$$

As expected, the double pole implies that there should be states with non-positive norm. This follows from considerations analogous to the above. For $x_0 > y_0$ we have again (1)

$$\begin{aligned} \langle 0|\varphi(x)\varphi(y)|0\rangle &= \frac{1}{(2\pi)^3}\int d^3p \exp[i\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{y})]\frac{d}{dp_0}\cdot \\ &\cdot\left[\frac{R_\lambda(p^2)}{(p_0 + \varepsilon)^2}\exp[-ip_0(x_0 - y_0)]\right]_{p_0=\varepsilon} + \ldots = \\ &= \int\frac{d^3p}{2\varepsilon}\left[\langle 0|\varphi(x)|g(\boldsymbol{p})\rangle\langle d(\boldsymbol{p})|\varphi(y)|0\rangle + \langle 0|\varphi(x)|d(\boldsymbol{p})\rangle\langle g(\boldsymbol{p})|\varphi(y)|0\rangle\right] + \ldots \end{aligned} \tag{2.9}$$

with

$$\begin{cases} \langle 0|\varphi(x)|g(\boldsymbol{p})\rangle = \dfrac{1}{(2\pi)^{\frac{3}{2}}}\exp[-ipx]\,, \\ \langle 0|\varphi(x)|d(\boldsymbol{p})\rangle = \dfrac{1}{(2\pi)^{\frac{3}{2}}}\left[\dfrac{R_\lambda(\lambda^2)\, ix_0}{2\varepsilon} + \dfrac{R_\lambda(\lambda^2)}{4\varepsilon^2} - \dfrac{R'_\lambda(\lambda^2)}{4\varepsilon}\right]\exp[-ipx]\,. \end{cases} \tag{2.10}$$

The wave function of g fulfils the Klein-Gordon equation (*i.e.* $|g(\boldsymbol{p})\rangle^{\pm}$ are eigenstates of the Hamiltonian); this is not true for d. The dipole ghost d continuously produces good ghosts g because of the x_0 term in (2.10). This is the reason for the notation good and bad ghosts.

The formula for g analogous to formula (2.5) is, as one can show by (2.7) and (2.10),

$$|g(f),\psi\rangle^{\pm} = \text{w-}\lim i\int \mathrm{d}^3x \left[\frac{\Box+\lambda^2}{-R_\lambda(\lambda^2)}\,\varphi(x)\right]\overleftrightarrow{\partial_0} f(x)|\psi\rangle^{\pm}\,. \tag{2.11}$$

For the reduction formula we get

$$\begin{aligned} {}^{-}\langle\psi', g(f)|\psi\rangle^{+} &= i\int \mathrm{d}^4x\, f^*(x)\left[\frac{(\Box+\lambda^2)^2}{-R_\lambda(\lambda^2)}\right]{}^{-}\langle\psi'|\varphi(x)|\psi\rangle^{+} = \\ &= -i\int \mathrm{d}^4p\, \tilde{f}^*(p)\,\frac{1}{\Delta(p)}\,\frac{1}{(2\pi)^2}\int \mathrm{d}^4x \exp[ipx]\,{}^{-}\langle\psi'|\varphi(x)|\psi\rangle^{+} \end{aligned} \tag{2.12}$$

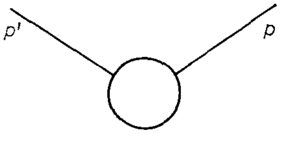
with $\tilde{f}^*(p)\sim\delta(p^2-\lambda^2)$, if $|\psi\rangle^+$ contains no $d(f)$.

Because of the nondiagonal metric (2.8) eq. (2.12) determines the probability that ψ, after scattering, is a product of ψ' and the dipole ghost $d(f)$.

There are corresponding but more complicated formulae for d. Since we define « physical states » by the condition that they contain no dipole ghosts before and after scattering, we do not need these formulae.

3. – Elastic scattering in an external field.

The S-matrix may be given by the diagram



with the propagator $\Delta(p)$ as external lines. $\Delta(p)$ may have the properties of Sect. **2**: a pole at $p^2=m^2$ with the state n and a double pole at $p^2=\lambda^2$ to which the ghosts g and d belong. The S-matrix may be pseudounitary, *i.e.* $S^\dagger S = SS^\dagger = 1$ in the state space with indefinite norm. Conservation of probability does not follow from this property, if there are transitions from n to g and d.

We get a unitary S-matrix if we construct « physical » states $\bar{n}$ with the condition that there are no dipole ghosts both before and after scattering. The unitarization method is uniquely defined by this condition. The physical states are superpositions of n and g. Since we must superimpose eigenstates of the S-operator, we first diagonalize the S-matrix. Formally we write

$${}^{-}\langle n(\boldsymbol{p}')|n(\boldsymbol{p})\rangle^{+} = \sum_r \langle n(\boldsymbol{p}')|n_r\rangle^{n}S_r^{n}\langle n_r|n(\boldsymbol{p})\rangle \tag{3.1}$$

with ${}^{-}\langle n_r|n_r\rangle^{+} = {}^{n}S^{n}_{r}$. Later we will perform the diagonalization explicitly. The variable r remains fixed in the following considerations and will be omitted. The index n denotes the in- (respectively out-) going state n. Correspondingly for g

$$\left\{\begin{array}{ll} {}^{-}\langle n|n\rangle^{+} = {}^{n}S^{n}\,, & {}^{-}\langle n|g\rangle^{+} = {}^{n}S^{g}\,, \\ {}^{-}\langle g|n\rangle^{+} = {}^{g}S^{n}\,, & {}^{-}\langle g|g\rangle^{+} = {}^{g}S^{g}\,. \end{array}\right. \tag{3.2}$$

In the (one-particle) space

$$\mathscr{H} = \mathscr{H}^{+}_{n} \oplus \mathscr{H}^{+}_{g} \oplus \mathscr{H}^{+}_{d} \tag{3.3}$$

the metric tensor has the form

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1\,, \\ 0 & 1 & 0 \end{matrix} \tag{3.4}$$

where the rows and columns refer respectively to $|n\rangle^{+}$, $|g\rangle^{+}$, $|d\rangle^{+}$. The pseudo-unitarity means

$$|n\rangle^{+\,+}\langle n| + |g\rangle^{+\,+}\langle d| + |d\rangle^{+\,+}\langle g| = 1\,. \tag{3.5}$$

Similarly for out states $|\,\rangle^{-}$.

We now define « physical » states $\bar{n}$ as a superposition of n and g:

$$|\bar{n}\rangle^{+} = |n\rangle^{+} + \varrho|g\rangle^{+} \tag{3.6}$$

with the condition that no dipole ghosts are emitted:

$${}^{-}\langle g|\bar{n}\rangle^{+} = 0\,. \tag{3.7}$$

Hence the space of physical in states is

$$\bar{\mathscr{H}}^{+} = (\mathscr{H}^{+}_{n} \oplus \mathscr{H}^{+}_{g}) \cap \mathscr{H}^{-\perp}_{g}\,. \tag{3.8}$$

The norm of $|\bar{n}\rangle^{+}$ is one by (3.3) and (3.5)

$${}^{+}\langle\bar{n}|\bar{n}\rangle^{+} = 1\,. \tag{3.9}$$

The out states $|n\rangle$ are defined correspondingly. Therefore we get the physical S-matrix

$${}^{-}\langle\bar{n}|\bar{n}\rangle^{+} = {}^{n}\bar{S}^{n} = \frac{\begin{vmatrix} {}^{n}S^{n} & {}^{n}S^{g} \\ {}^{g}S^{n} & {}^{g}S^{g} \end{vmatrix}}{{}^{g}S^{g}}\,. \tag{3.10}$$

It is easy to prove the unitarity of $\bar{S}$ by using (3.5), (3.6), (3.7) and (3.9)

$$|{}^{n}\bar{S}^{n}|^2 = {}^{+}\langle\bar{n}|\bar{n}\rangle^{-}\,{}^{-}\langle\bar{n}|\bar{n}\rangle^{+} = {}^{+}\langle\bar{n}|n\rangle^{-}\,{}^{-}\langle n|\bar{n}\rangle^{+} = \tag{3.11}$$
$$= {}^{+}\langle\bar{n}|\big[|n\rangle^{-}\,{}^{-}\langle n| + |g\rangle^{-}\,{}^{-}\langle d| + |d\rangle^{-}\,{}^{-}\langle g|\big]|\bar{n}\rangle^{+} = {}^{+}\langle\bar{n}|\bar{n}\rangle^{+} = 1\,.$$

If $\mathscr{H}$ were positive definite, the unitarity (3.11) would imply

$$\bar{\mathscr{H}}^{+} = \bar{\mathscr{H}}^{-}\,. \tag{3.12}$$

This is not true in general if $\mathscr{H}$ is indefinite as in our case. But because of the special metric (3.4) and the special unitarization this equation holds here too. From (3.4) and (3.8) it follows that

$$\bar{\mathscr{H}}^{+} = (\mathscr{H}_{n}^{+} \oplus \mathscr{H}_{g}^{+}) \cap \mathscr{H}_{g}^{-\perp} = \mathscr{H}_{g}^{+\perp} \cap \mathscr{H}_{g}^{-\perp} = \bar{\mathscr{H}}^{-} = \bar{\mathscr{H}}\,. \tag{3.13}$$

If

$$P = |\bar{n}\rangle^{+}\,{}^{+}\langle\bar{n}| = |\bar{n}\rangle^{-}\,{}^{-}\langle\bar{n}| \tag{3.14}$$

is the projector on $\bar{\mathscr{H}}$, we get for the physical S-matrix

$$\bar{S} = PSP \tag{3.15}$$

and

$$\bar{S}^{\dagger}\bar{S} = \bar{S}\bar{S}^{\dagger} = P\,. \tag{3.16}$$

However, $\bar{\mathscr{H}}$ is generally not an invariant subspace under S. But the proportion of good ghosts in $|\bar{n}\rangle^{+}$ is the same before and after scattering up to a phase factor, if time inversion invariance is assumed:

$$|{}^{+}\langle d|\bar{n}\rangle^{+}|^2 = {}^{+}\langle d|\bar{n}\rangle^{+}\,{}^{+}\langle\bar{n}|d\rangle^{+} = {}^{+}\langle d|\bar{n}\rangle^{-}\,{}^{-}\langle\bar{n}|d\rangle^{+} = |{}^{-}\langle\bar{n}|d\rangle^{+}|^2 = |{}^{-}\langle d|\bar{n}\rangle^{+}|^2\,. \tag{3.17}$$

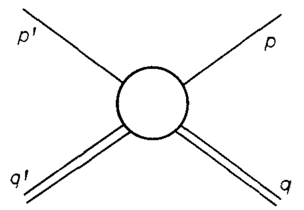
Hence ${}^{+}\langle d|\bar{n}\rangle^{+}$ and ${}^{-}\langle d|\bar{n}\rangle^{+}$ only differ by a phase factor which is generally different from one.

4. – Elastic scattering of two particles.

4·1. *Partial-wave expansion.* – Let us consider two real scalar fields $\varphi(x)$ and $\phi(y)$ with the propagators $\Delta_m(p)$ respectively $\Delta_M(q)$. $\Delta_m(p)$ may have the same properties as in Sect. **3**, *i.e.* a pole at $p^2 = m^2$ with the state n and a double pole at $p^2 = \lambda^2$ with the ghosts g and d. $\Delta_M(q)$ may have a simple pole at $q^2 = M^2$ associated with the particle N. We consider the elastic scattering

$$n(\boldsymbol{p}) + N(\boldsymbol{q}) \to n(\boldsymbol{p}') + N(\boldsymbol{q}')$$

and the diagram



with the total momentum $P = p + q = p' + q'$ $(P^2 = s)$.

From (2.6) and (2.12) we get for the scattering amplitude

$$ {}^{p'}_{q'}T^{p}_{q}(s, t) = \frac{1}{\Delta_m(p')\Delta_M(q)}\, \tilde{\tau}(p', q', p, q)\, \frac{1}{\Delta_m(p)\Delta_M(q)} \,. \tag{4.1}$$

$\tilde{\tau}$ is the time-ordered 4-point function in momentum space. T is a function of p'^2, q'^2, p^2, q^2, s, t. To diagonalize T we take the centre-of-mass system $\boldsymbol{P} = 0$. Then

$$ \left\{ \begin{aligned} t &= (p'-p)^2 = p'^2 + p^2 - 2p'_0 p_0 + 2|\boldsymbol{p}'||\boldsymbol{p}|\hat{\boldsymbol{p}}'\hat{\boldsymbol{p}} \,, \\ p_0 &= \frac{1}{2\sqrt{s}}(s + p^2 - q^2) \,, \qquad |\boldsymbol{p}| = \frac{1}{2\sqrt{s}}\sqrt{\lambda(s, p^2, q^2)} \end{aligned} \right. \tag{4.2}$$

(with $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$); $\hat{\boldsymbol{p}}$ is the unit vector parallel to $\boldsymbol{p}$. The partial wave expansion for T is

$$ {}^{p'}_{q'}T^{p}_{q}(s, t) = 4\pi \sum_{l,m} {}^{p'}_{q'}t^{p}_{q}(s, l)\, Y^*_{lm}(\hat{\boldsymbol{p}}')\, Y_{lm}(\hat{\boldsymbol{p}}) \,. \tag{4.3}$$

The partial-wave amplitude t is a function of the masses of the external lines and the conserved quantities s and l. The different matrix elements are analogous to (3.2)

$$ {}^{n}_{N}t^{n}_{N} \,, \qquad {}^{n}_{N}t^{g}_{N} \,, \qquad {}^{g}_{N}t^{n}_{N} \,, \qquad {}^{g}_{N}t^{g}_{N} \,, $$

if the indices denote the respective states n, g and N.

The « physical » partial-wave amplitude $\bar{t}$ is corresponding to (3.10)

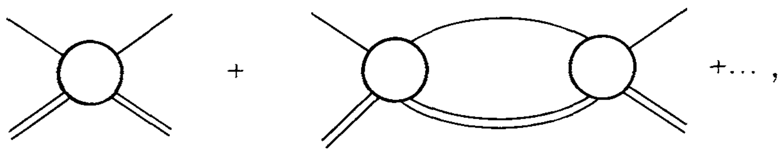
$$ {}^{n}_{N}\bar{t}^{n}_{N} = \frac{\begin{vmatrix} {}^{n}_{N}t^{n}_{N} & {}^{n}_{N}t^{g}_{N} \\ {}^{g}_{N}t^{n}_{N} & {}^{g}_{N}t^{n}_{N} \end{vmatrix}}{{}^{g}_{N}t^{g}_{N}} \,, \tag{4.4}$$

and the « physical » scattering amplitude $\bar{T}$ is given by (4.3), substituting $T \to \bar{T}$ and $t \to \bar{t}$. $\bar{S}$ is unitary according to Sect. **3**.

A local coupling as for example $\mathscr{L}_I \sim \varphi^2 \phi^2$ implies that ${}^{p'}_{q'}T^{p}_{q} = \text{const}$ in first order. Hence $\bar{t}$ and $\bar{T}$ vanish by (4.4) in first order. In general $\bar{t}$ vanishes if ${}^{p'}_{q'}t^{p}_{q}$ is a product $f(p'^2, q'^2)\cdot f(p^2, q^2)$.

This fact means that, in a certain manner, the « local part » of the interaction does not contribute to $\bar{T}$.

4·2. « *Wrong cuts* ». – We consider a perturbation expansion for T



(4.5)
$$
{}^{p'}_{q'}T^{p}_{q} = {}^{p'}_{q'}F^{p}_{q} + \int \mathrm{d}^4k\, {}^{p'}_{q'}F^{k}_{l}\, \Delta_m(k)\, \Delta_M(l)\, {}^{k}_{l}F^{p}_{q} + \dots,
$$

with $l = P - k$. T is singular, if two singularities of Δ_m and Δ_M coincide and the integration path runs between both. Hence T has a physical branch cut at $s \geqslant (m + M)^2$ arising from the poles of Δ_m and Δ_M at $k^2 = m^2$ respectively, $l^2 = M^2$ according to the intermediate states nN and a « wrong cut » at $s \geqslant (\lambda + M)^2$ arising from the double pole of Δ_m at $k^2 = \lambda^2$ and the pole of Δ_M at $l^2 = M^2$ according to the ghost intermediate states gN and dN. This cut contradicts unitarity, therefore it drops out if we adopt the unitarization as in (4.4). That shall be shown for (4.5), to get a deeper insight as to how this unitarization method actually works.

We put into (4.5) a partial-wave expansion for F analogous to (4.3). In the centre-of-mass system ($\boldsymbol{P} = 0$) $l^2 = (P - k)^2$ does not depend on $\hat{k}$. Hence we can perform the $\hat{k}$-integration and obtain for the partial-wave expansion by means of the orthogonality of the spherical functions

$$
{}^{p'}_{q'}t^{p}_{q} = {}^{p'}_{q'}f^{p}_{q} + 4\pi \int \mathrm{d}k_0\, k^2\, \mathrm{d}k\, {}^{p'}_{q'}f^{k}_{l}\, \Delta_m(k)\, \Delta_M(l)\, {}^{k}_{l}f^{p}_{q} + \dots . \tag{4.6}
$$

We now put this equation into the determinant formula (4.4) and get for the physical partial-wave amplitude after some calculations and neglecting terms of higher order

$$
{}^{n}_{N}\bar{t}^{n}_{N} = {}^{n}_{N}\bar{f}^{n}_{N} + \int \mathrm{d}^4 k\, {}^{n}_{N}\bar{f}^{k}_{l}\, \Delta_m(k)\, \Delta_M(l)\, {}^{k}_{l}\bar{f}^{n}_{N} + \dots . \tag{4.7}
$$

The functions ${}^{n}_{N}\bar{f}^{k}_{l}$ etc. are defined by the continuation of (4.4):

$$
{}^{p'}_{q'}\bar{f}^{p}_{q} = \frac{\begin{vmatrix} {}^{p'}_{q'}f^{p}_{q} & {}^{p'}_{q'}f^{g}_{N} \\ {}^{g}_{N}f^{p}_{q} & {}^{g}_{N}f^{g}_{N} \end{vmatrix}}{{}^{g}_{N}f^{g}_{N}} . \tag{4.8}
$$

We again use the following notation: if f^p is a function of p^2, we write $f^n = f^p|_{p^2=m^2}$, $f^g = f^p|_{p^2=\lambda^2}$ (similarly for the other variables).

The « physical vertex functions » ${}^{n}_{N}\bar{f}^{k}_{l}$ and ${}^{k}_{l}\bar{f}^{n}_{N}$ vanish for $k^2 = \lambda^2$ and $l^2 = M^2$, since both columns (respectively rows) of the determinants become equal. These two zeros removes the « wrong cut » which arises from the double pole of $\Delta_m(k)$ at $k^2 = \lambda^2$ at the simple poles of $\Delta_M(l)$ at $l^2 = M^2$, as was expected by unitarity arguments.

5. – Bound states.

The same assumptions as in Sect. **4** are assumed to hold. Let Ψ be a bound state with the Bethe-Salpeter amplitude

$$\langle 0|T\varphi(x)\phi(y)|\Psi\rangle = \frac{1}{(2\pi)^4}\int \mathrm{d}^4p'\,\mathrm{d}^4q'\,\exp[-ixp'-iyq']\,{}^{p'}_{q'}\Psi\,. \tag{5.1}$$

${}^{p'}_{q'}\Psi$ is to fulfil the Bethe-Salpeter equation

$$ {}^{p'}_{q'}\Psi = \Delta_m(p')\,\Delta_M(q')\int \mathrm{d}^4k\,{}^{p'}_{q'}F^{k}_{l}\,{}^{k}_{l}\Psi \tag{5.2}$$

with $l = P - k$, $P = p' + q'$. There arise analogously to (4.5) at $s \geqslant (m+M)^2$ a physical and at $s \geqslant (\lambda + M)^2$ a wrong cut. If the mass $\sqrt{s}$ of Ψ is below both thresholds, then (5.2) may be solved in the usual manner.

But there also should exist bound states with $\lambda + M < \sqrt{s} < m + M$, since they cannot decay into physical particles nN, but only into ghosts which cannot be detected experimentally. In this case Ψ contains a part Ψ_0 of « free » good ghosts gN, *i.e.* according to (2.4) and (2.10)

$$ {}^{p'}_{q'}\Psi^{p}_{0q} = {}^{p'}_{q'}\Psi^{g}_{0N} = \varrho\,\delta^{(4)}(p'-p)\,\delta^{(4)}(q'-q) \qquad \text{with } p^2 = \lambda^2,\ q^2 = M^2\,. \tag{5.3}$$

Hence we obtain in place of (5.2) for $\lambda + M < \sqrt{s} < m + M$ after transition to partial waves analogous to (4.6) the integral equation

$$ {}^{p'}_{q'}\psi = \Delta_m(p')\,\Delta_M(q')\int \mathrm{d}^4k\,{}^{p'}_{q'}f^{k}_{l}\,{}^{k}_{l}\psi + {}^{p'}_{q'}\psi^{g}_{0N}\,. \tag{5.4}$$

(The energy and spin variables are suppressed.)

Usually such an inhomogeneous equation has no normalizable solutions belonging to discrete eigenvalues, but there are for all energies improper, non-normalizable, scattering states. However, to get a « physical » state $\bar{\psi}$, we must respect the conditions (3.7) that no dipole ghosts dN are emitted:

$$ {}^{-}\langle {}^{g}_{N}|\bar{\psi}\rangle^{+} = 0\,. \tag{5.5}$$

By means of the reduction formulae (2.6) and (2.12) and eq. (5.4) we obtain

$$\int \mathrm{d}^4 k \, {}^{g}_{N} f^{k\,k}_{l\;l} \bar{\psi} = 0 \,. \tag{5.6}$$

Under these conditions the integral equation (5.4) can have a normalizable solution to a discrete eigenvalue for $\lambda + M < \sqrt{s} < m + M$, since from (5.4) and (5.6) we get the homogeneous equation for $\bar{\chi} = \bar{\psi} - \psi_0$

$$ {}^{p'}_{q'}\bar{\chi} = \Delta_m(p')\, \Delta_M(q') \int \mathrm{d}^4 k \, {}^{p'}_{q'}\bar{f}^{k\,k}_{l\;l} \bar{\chi} \,. \tag{5.7}$$

For ${}^{p'}_{q'}\bar{f}^{k\,k}_{l\;l}\psi_0$ vanishes because of (5.3) and (4.8). The state $|\bar{\psi}\rangle$ is normalizable, although it contains the « free » ghost part $|\psi_0\rangle$. The factor ϱ in (5.3) can be calculated by the normalization condition $\langle\bar{\psi}|\bar{\psi}\rangle = 1$ and (5.6).

6. – Generalization and discussion.

We obtained for a given pseudounitary S-matrix S in a state space $\mathscr{H}$ with indefinite metric by means of a unique unitarization procedure an S in a definite state space $\bar{\mathscr{H}}$ ([*]). The methods of this paper can be generalized to particles with spin and other internal degrees of freedom. But the diagonalization of the S-matrix is more complicated. The unitarization is always applicable, if the metric is indefinite due only to dipole ghosts.

The general formulation of this unitarization method is:

Let $\mathscr{H}_N$, $\mathscr{H}_G$ and $\mathscr{H}_D$ be the Fock spaces of the states with positive norm respectively ghost respectively dipole ghost states associated with the simple respectively double poles of the propagators. Let $\mathscr{H}'_G$ be equal to $\mathscr{H}_G/\{|0\rangle\}$. Then the space of physical incoming N-states is

$$\bar{\mathscr{H}}^{+}_{N} = (\mathscr{H}^{+}_{N} \oplus \bar{\mathscr{H}} \oplus \mathscr{H}'_G) \cap (\bar{\mathscr{H}} \oplus \mathscr{H}'^{-}_{G})^{\perp} \,.$$

This is an implicit equation for $\bar{\mathscr{H}}$, which can be solved in certain approximations. If the states contain at most one ghost $\bar{\mathscr{H}}^{+}_{N} = \bar{\mathscr{H}}^{-}_{N} = \bar{\mathscr{H}}$. In general, however, there are transitions to new states with positive norm ([6]) associated with the double poles.

* * *

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([*]) If this space still contains zero vectors, one must take the factor space over the radical, as *e.g.* in QED.

● RIASSUNTO (*)

Si considera una matrice S definita da diagrammi di Feynman in uno spazio degli stati con norma indefinita. I propagatori possono avere, oltre che poli semplici, anche poli doppi nello spazio degli impulsi. Si deduce un procedimento di unitarizzazione che proietta la matrice S nel sottospazio fisico.

(*) *Traduzione a cura della Redazione.*

Дипольные « духи » и унитарность.

Резюме (*). — Рассматривается S-матрица, определенная диаграммами Фейнмана в пространстве состояний с индефинитной нормой. Пропагаторы могут иметь, помимо простых полюсов, также двойные полюса в импульсном пространстве. Выводится процедура унитаризации, которая проектирует S-матрицу на физическое подпространство.

(*) *Переведено редакцией.*