On the completeness of the Bethe-states for the supersymmetric t-J model

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Abstract

A complete set of eigenstates for the supersymmetric t-J model in one dimension is obtained by combining the Bethe-ansatz with the Spl(2,1) supersymmetry of the model.

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Since Bethe's[1] work on the isotropic Heisenberg model, it is known that his famous Bethe-ansatz alone does not provide a complete set of eigenvectors, but it only determines the highest weight vectors of multiplets of the underlying SU(2) symmetry group (see also ref.2). Recently Essler, Korepin and Schoutens showed that for the one dimensional Hubbard model[3] the Bethe-ansatz states are the highest (or lowest) weight vectors of a complete system of SO(4) multiplets. In this letter, we solve for the first time the completeness problem of the Bethe-ansatz for a supersymmetric integrable model.

The t-J model has attracted a great interest in connection with high-Tc superconductivity. This model, proposed by Anderson [4], is obtained from the Hubbard model by excluding two electrons at one site. It describes fermions with nearest-neighbour hopping and spin exchange interaction on a lattice. For an appropriate choice of the coupling constants the model becomes supersymmetric[5] and its eigenstates are given by irreducible representations of the Spl(2,1) superalgebra, which was investigated by Scheunert, Nahm and Rittenberg[6]. For the one dimensional case the t-J model is exactly solvable by the nested Bethe-ansatz[7,8,9]. We will show that the Bethe-ansatz together with the supersymmetry of the model yield to a complete set of eigenvectors.

The Hamiltonian of the t-J model for a one-dimensional lattice of L sites is[4]

$$\mathcal{H} = P\{-t\sum_{j,\sigma}(c^\dagger_{j,\sigma}c_{j+1,\sigma} + c^\dagger_{j+1,\sigma}c_{j,\sigma})\}P + J\sum_j(\vec{S}_j.\vec{S}_{j+1} - \frac{n_j n_{j+1}}{4}), \qquad (1)$$

where the projector $P = \prod_{j=1}^{L}(1 - n_{j\uparrow}n_{j\downarrow})$ restricts the Hilbert space by the constraint of no double occupancy. This Hamiltonian can be rewritten in terms of Hubbard's projection operators[5,10]

$$X_j^{\alpha\beta} = |\alpha_j><\beta_j|; \qquad \alpha,\beta = 0,1,2\ , \qquad (2)$$

where $|0_j>$ denotes a hole and $|a_j>$ an electron with spin up ($a = 1$) or down ($a = 2$) at site j. The operators X_j^{a0} and X_j^{0a} are fermionic, since they create or annihilate an electron with spin a, whereas X_j^{ab} and X_j^{00} are bosonic. Using (2) the Hamiltonian up to a chemical potential reads

$$\mathcal{H} = -t\sum_{a,j}(X_j^{a0}X_{j+1}^{0a} + X_{j+1}^{a0}X_j^{0a}) + \frac{J}{2}\sum_j\left(\sum_{a,b}X_j^{ab}X_{j+1}^{ba} - X_j^{00}X_{j+1}^{00}\right). \qquad (3)$$

The symmetries of this Hamiltonian are U(1) gauge, SU(2) spin and lattice translational invariance. Moreover, the model becomes supersymmetric at $J = 2t$ [5]. The operators $X^{0a} = \sum_j X_j^{0a}$, $X^{a0} = \sum_j X_j^{a0}$ and $X^{ab} = \sum_j X_j^{ab}$ are the generators of the superalgebra Spl(2,1)[6,8]

$$[X^{\alpha\beta}, X^{\alpha'\beta'}]_\pm = X^{\alpha\beta'}\delta_{\beta\alpha'} \pm \delta_{\beta'\alpha}X^{\alpha'\beta}, \tag{4}$$

where $\pm$ is chosen according to Fermi or Bose statistics. For $J = 2t$ the Hamiltonian is invariant with respect to this superalgebra, i.e.,

$$[\mathcal{H}, X^{0a}] = [\mathcal{H}, X^{a0}] = [\mathcal{H}, X^{ab}] = 0. \tag{5}$$

Furthermore, in this case the model is exactly solvable by the nested Bethe-ansatz. According to this method, the spectrum of the Hamiltonian is written in terms of parameters $\{v_j\}, j = 1, \ldots, N$ and $\{\gamma_\alpha\}, \alpha = 1, \ldots, M$. N is the number of holes plus down spins and M is the number of holes. The parameters v_j and γ_α are solutions of the Bethe-ansatz equations, a system of coupled algebraic equations[7,8,11,12]

$$\begin{gathered}\left(\frac{v_j + i}{v_j - i}\right)^L = -\prod_{k=1}^{N}\left(\frac{v_j - v_k + 2i}{v_j - v_k - 2i}\right)\prod_{\beta=1}^{M}\left(\frac{v_j - \gamma_\beta - i}{v_j - \gamma_\beta + i}\right) \\ \prod_{j=1}^{N}\left(\frac{\gamma_\alpha - v_j + i}{\gamma_\alpha - v_j - i}\right) = 1.\end{gathered} \tag{6}$$

In addition, the following inequalities hold[11]

$$\begin{gathered}M \le N \le \frac{L+M}{2} \\ 0 \le M \le L.\end{gathered} \tag{7}$$

The physical content of (7) is that the magnetization $Sz = \frac{1}{2}(n_\uparrow - n_\downarrow) = \frac{1}{2}(L - 2N + M)$ and the filling $F = \frac{n_\uparrow + n_\downarrow}{n_\uparrow + n_\downarrow + n_h} = 1 - \frac{M}{L}$ are restricted to $0 \le Sz \le \frac{L}{2}$ and $0 \le F \le 1$, respectively. Here n_h is the number of holes. The basic procedure to solve eqs. (6) is to adopt the string-conjecture[11,13], which means in this case that all roots γ's are real and the v's appear as strings

$$\begin{aligned}&\gamma_\beta = real; \qquad \beta = 1, \ldots, M \\ &v_{\alpha j}^n = v_\alpha^n + i(n + 1 - 2j); \quad j = 1, \ldots, n; \quad \alpha = 1, \ldots, N_n; \quad n = 1, 2, \ldots,\end{aligned} \tag{8}$$

where v_α^n is the position of the center of the string on the real v-axis and the number of n-strings N_n satisfy the relation

$$N = \sum_n n N_n. \tag{9}$$

Inserting this conjecture in (6) and taking its logarithm we obtain the coupled equations for the v_α^n and γ_β

$$L\theta\left(\frac{v_\alpha^n}{n}\right) - \underbrace{\sum_m \sum_{\beta=1}^{N_m}}_{(m,\beta)\neq(n,\alpha)} \Theta_{nm}(v_\alpha^n - v_\beta^m) + \sum_{\beta=1}^{M} \theta\left(\frac{v_\alpha^n - \gamma_\beta}{n}\right) = 2\pi I_\alpha^n$$

$$\sum_n \sum_{\alpha=1}^{N_n} \theta\left(\frac{v_\alpha^n - \gamma_\beta}{n}\right) = 2\pi J_\beta, \tag{10}$$

where

$$\theta(x) = 2\arctan x$$

$$\Theta_{nm}(x) = \begin{cases} \theta(\frac{x}{|n-m|}) + 2\theta(\frac{x}{|n-m|+2}) + \ldots + 2\theta(\frac{x}{n+m-2}) + \theta(\frac{x}{n+m}) & \text{for } n \neq m \\ 2\theta(\frac{x}{2}) + 2\theta(\frac{x}{4}) + \ldots + 2\theta(\frac{x}{2n-2}) + \theta(\frac{x}{2n}) & \text{for } n = m. \end{cases} \tag{11}$$

So the solutions of (6) are parametrized in terms of the numbers I_α^n and J_β. I_α^n is an integer (half-integer) if $L + M - N_n$ is odd (even) and J_β is an integer (half-integer) if $\sum_n N_n$ is even (odd). In addition they are limited to the intervals

$$|I_\alpha^n| \leq \frac{1}{2}(L + M - \sum_m t_{nm} N_m - 1)$$

$$|J_\beta| \leq \frac{1}{2}(\sum_n N_n - 2), \tag{12}$$

where $t_{nm} = 2min(n,m) - \delta_{nm}$. In fact, all sets $\{I_\alpha^n, J_\beta\}$ where the I's and J's are pairwise different specify all the Bethe vectors ($|\psi_{Bethe} >_{N,M}$), which are mutually orthogonal. A configuration with coinciding I's or J's would lead to a vanishing eigenfunction. Therefore, the number of existing Bethe vectors for fixed N, M is given by

$$Z(N,M) = \sum_{\{N_n\}} \binom{q-1}{M} \prod_n \binom{L - \sum_m t_{nm} N_m + M}{N_n}, \tag{13}$$

where $q = \sum_n N_n$ and the sum over $\{N_n\}$ is constrained to $\sum_n nN_n = N$. It is convenient to write this sum as

$$Z(N,M) = \sum_{q=0}^{N} \binom{q-1}{M} \sum_{\{N_n\}} \prod_n \binom{L - \sum_m t_{nm} N_m + M}{N_n}, \tag{14}$$

where the inner sum is constrained to fixed values of N and q. This expression resembles the one calculated by Bethe in the isotropic Heisenberg model[1,2]. The above equation can be simplified to

$$Z(N,M) = \sum_{q=0}^{N} \frac{L+M-2N+1}{L+M-N+1} \binom{q-1}{M} \binom{L+M-N+1}{q} \binom{N-1}{q-1}. \tag{15}$$

The total number of Bethe vectors is obtained by summing $Z(N,M)$ over all N,M restricted to (7). However, this number is less than 3^L, so that the Bethe-ansatz does not yield all the eigenvectors of the model. In fact, this feature is typical and appears in other integrable models, e.g. the isotropic Heisenberg model[1,2] and the Hubbard model[3]. In order to construct a complete set we shall invoke the supersymmetry of the Hamiltonian. First, from (5) follows that the eigenvectors are classified by multiplets corresponding to irreducible representations of the superalgebra Spl(2,1). In addition, we can show[11] that Bethe-ansatz vectors are highest weight vectors, i.e.,

$$X^{\alpha\beta} |\Psi_{Bethe} >_{N,M} = 0, \qquad \alpha < \beta. \tag{16}$$

Then by acting with the Spl(2,1) lowering operators ($X^{\alpha\beta}, \alpha > \beta$) on the Bethe states we obtain additional eigenvectors. So, each Bethe vector (with fixed N,M in the interval (7)) is the highest weight vector in a multiplet of dimension[6,11]

$$d(N,M) = \begin{cases} 4S_z + 1 = 2L + 1 & \text{if } N = M = 0 \\ 8(S_z + 1/2) = 4(L - 2N + M + 1) & \text{otherwise.} \end{cases} \tag{17}$$

With these considerations, the full number of eigenstates of the Hamiltonian is

$$
\begin{aligned}
Z &= \sum_{M=0}^{L} \sum_{N=M}^{\frac{L+M}{2}} d(N,M) Z(N,M) \\
&= 2L+1+4\sum_{N=1}^{\frac{L}{2}} (L-2N+1)\frac{L-2N+1}{L-N+1}\sum_{q=1}^{N}\binom{L-N+1}{q}\binom{N-1}{q-1} \\
&\quad +4\sum_{M=1}^{L}\sum_{N=M}^{\frac{L+M}{2}}(L-2N+M+1)\frac{L+M-2N+1}{L+M-N+1}\cdot \\
&\quad \sum_{q=1}^{N}\binom{q-1}{M}\binom{L+M-N+1}{q}\binom{N-1}{q-1}.
\end{aligned}
\tag{18}
$$

The first sum in (18) can be performed (see ref. 2). The second sum is more complicated. After a cumbersome calculation (details are given in ref.11) we arrive at

$$
\begin{aligned}
Z &= 2L+1+2^{L+2}-4(L+1)+\frac{4L!}{(L-2)!}\int_0^1 (1-p)\left[(1+2p)^{L-2}-(1+p)^{L-2}\right]dp \\
&= 3^L.
\end{aligned}
\tag{19}
$$

So, we showed that the number of orthogonal eigenvectors of the t-J model is 3^L, which is precisely the number of states in the Hilbert space of a chain of length L, where at each site there may be either a spin up or a spin down electron or a hole.

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