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State sum invariants of compact 3-manifolds with boundary and 6*j*-symbols

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Abstract. We extend the combinatorial construction of invariants of smooth, compact, closed 3-manifolds as given by Turaev and Viro to obtain invariants of 3-manifolds with boundary. The technique uses quantum $6j$-symbols associated to the quantized universal enveloping algebra $\mathrm{U}_q(\mathrm{sl}(2,\mathbb{C}))$ (q a root of unity) to give a construction of a state sum for given triangulation. This state sum is then invariant under subdivisions and isotopies of the boundary. Our methods also lead to a simplified proof of the main result of Turaev and Viro that the state sum is independent of the triangulation and hence gives rise to an invariant of the manifold. We use surgery to calculate the state sum for some (closed) manifolds. Our results also confirm a recent finding of Turaev, which relates this theory to the topological quantum field theory with a Chern–Simons action in the sense of Witten.

1. Introduction

In an article Turaev and Viro [16] have constructed non-trivial 'quantum' invariants of compact 3-manifolds M in the form of state sums (called partition functions in statistical physics and vacuum functionals in quantum field theory) associated with the quantized universal enveloping algebra $\mathrm{U}_q(\mathrm{sl}(2,\mathbb{C}))$, where q is a complex root of unity of a certain degree $2r > 4$. The state sum is first defined for a given triangulation X of M and then shown to be independent of the triangulation thus giving rise to a well defined invariant of M, which thus depends on q. This result may be viewed as a rigorous mathematical construction of what is called topological quantum field theory (see also [4]). In fact, in the language of physicists, a triangulation corresponds to the introduction of a high-energy cut-off. Now topological quantum field theories have trivial dynamics, are scale-invariant and more generally independent of any metrics. Invariance under subdivisions is just the statement that the renormalization group transformation is trivial. This result suggests that the familiar techniques from algebraic topology should become useful to construct and discuss other topological quantum field theories.

The purpose of this article is to introduce observables into the Turaev–Viro approach in the form of certain closed (piecewise) smooth 2-submanifolds forming the boundary ∂M of M, whose 'expectation values' are invariants. In the case that the 2-submanifolds consists of several copies of 2-tori, these 2-tori may be viewed as blown-up links, i.e. the boundaries of tubular neighbourhoods of such links. Invariance is then just the statement that these 2-tori around the links may be chosen arbitrarily 'small'. In our construction of the state sum, ∂M is not assumed to be orientable, the case $M = \mathbf{R}P^2 \times [0,1]$, $\partial M = \mathbf{R}P^2 \times \{0\} \cup \mathbf{R}P^2 \times \{1\}$ being an example.

Now we briefly outline our approach. Let X be a triangulation of M which induces a triangulation ∂X of ∂M. Then to X we associate a state sum $Z(X)$ using in the construction $6j$-symbols within the abstract and more general set-up of Turaev and Viro, for which the quantum $6j$-symbols of $\mathrm{U}_q(\mathrm{sl}(2,\mathbf{C}))$ form an example. Our construction agrees with that in [16] for the case $\partial M = \emptyset$. For sufficiently fine triangulations (a notion which we will make precise below), we show that this state sum is independent of the triangulation and hence defines a state sum $Z(M)$. Moreover, we show that the state sum $Z(X)$ is invariant under simple isotopy of the boundary ∂X. We recall that basically a simple isotopy of ∂X consists of successive addition to X of 3-simplexes $\sigma^3 \in X$ which intersects ∂X in exactly r $(1 \leqslant r \leqslant 3)$ 2-simplexes in common with ∂X.

Roughly speaking the construction of the state sum $Z(X)$ goes as follows. As in [16] an edge colouring of X is a map $\sigma^1 \mapsto j(\sigma^1)$ from the set of 1-simplexes of X into a finite set I. This is a set of representations for the quantum group case. For suitable edge colourings, one associates to each 3-simplex σ^3 the $6j$-symbol

$$\begin{vmatrix} j(\sigma_1^1) & j(\sigma_2^1) & j(\sigma_3^1) \\ j(\sigma_4^1) & j(\sigma_5^1) & j(\sigma_6^1) \end{vmatrix} \tag{1.1}$$

where σ_k^1 and σ_{3+k}^1 $(k = 1,2,3)$ are opposite edges in σ^3. In addition we define a vertex colouring to be a map $J : \sigma^0 \mapsto J(\sigma^0)$ from the set of vertices of ∂X into I. For a suitable edge colouring $\underline{j}$ and suitable vertex colouring $\underline{J}$ one associates to each 2-simplex $\sigma^2 \in \partial X$ the $6j$-symbol

$$\begin{vmatrix} j(\sigma_1^1) & j(\sigma_2^1) & j(\sigma_3^1) \\ J(\sigma_1^0) & J(\sigma_2^0) & J(\sigma_3^0) \end{vmatrix} \tag{1.2}$$

where σ_k^0 are the vertices opposite to the edges σ_k^1 $(k = 1,2,3)$ of the 2-simplex σ^2. We multiply the $6j$-symbols (1.1) over all 3-simplexes σ^3 in X and the $6j$-symbols (1.2) over all 2-simplexes in ∂X. The resulting expression is multiplied by a certain weight (similar to the procedure in [16]). The state sum is then given by summing over all $\underline{j}$ and $\underline{J}$, for which the $6j$-symbols (1.1) and (1.2) are defined and non-zero. Note that in contrast to lattice gauge theories where the sum is taken over the group, here the sum is taken over the dual variables, namely the representations of the quantum group.

The invariance of the state sum under isotopies of the boundary is an easy consequence of five polynomial relations involving five $6j$-symbols, which follow from a Biedenharn–Elliot relation. In fact these relations can be viewed as combinatorial versions of a local Stokes theorem. It is remarkable that in the context of Regge calculus a Stokes-theorem-type relation is also responsible for the invariance of Lipschitz–Killing curvatures on PL spaces under subdivisions [10, 3]. Invariance under subdivisions is then a consequence of the Stokes theorem mentioned above and relies on the following arguments. Consider a subdivision of X localized in $Y \subset X$. There are two cases:

(i) If Y is near the boundary ∂X, we shift ∂X using the invariance under isotopies of ∂X such that the state sum has no contributions from Y.

(ii) If it is not near the boundary ∂X, using our Stokes theorem we create a hole close to Y and obtain again case (i). In fact by Stokes theorem the contributions from the interior of Y are replaced by a contribution on ∂Y.

One natural generalization of our approach would be to work with other q-Lie groups [8] (see also [4]). Secondly one may introduce on the 2-manifolds ∂M additional loops (links) L' which cross the magnetic flux lines. If the statistical weights of these crossings are given by R-matrices, the state sum is q-invariant and may be formulated analogously to Z†. Another interesting program is to consider the 'semiclassical' limit $q \to 1$. This could shed some new light on the observation in [9], which relates (classical) $6j$-symbols to Regge calculus.

In analogy to Turaev and Viro one can freeze the colouring on parts of ∂M in order to discuss cobordism theory. In this paper we do not discuss this issue.

The paper is organized as follows. In section 2 we define the state sum Z for 3-manifolds with boundary and prove invariance under isotopy of the boundary. We start with a given triangulation and prove independence under Alexander moves. In section 3 the state sum Z is calculated explicitly for some examples using surgery techniques.

2. Construction of a state sum

As announced in the introduction, in this section we will work within the general axiomatic set-up of Turaev and Viro. In this section we will generalize this construction and in the next section we will explicitly calculate the state sum for some examples.

For the convenience of the reader and in order to establish notation, we give a brief review of the set-up in [16]. Let K be a commutative ring with unit. By K^* we denote the set of invertible elements in K. Let I be a finite set, $w \in K^*$ a distinguished element and $i \mapsto w_i$ a map from I into K^*. We set

$$\tilde{w}^2 = \sum_{i \in I} w_i^4 . \tag{2.1}$$

We assume there is given a non-empty set of unordered triples $(i, j, k) \in I$ called admissible. We set $\delta(i, j, k) = 1$ if (i, j, k) is admissible and zero otherwise. An ordered 6-tuple (i, j, k, l, m, n) is called admissible if the four unordered 3-triples (i, j, k), $(k, l, m), (i, m, n)$ and (j, l, n) are admissible. To each such admissible 6-tuple we assume there is associated an element of K the abstract $6j$-symbol denoted by

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$$

satisfying the following symmetry relations:

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} j & i & k \\ m & l & n \end{vmatrix} = \begin{vmatrix} i & k & j \\ l & n & m \end{vmatrix} = \begin{vmatrix} i & m & n \\ l & j & k \end{vmatrix} . \tag{2.2}$$

† After completion of our calculations we received a preprint [14] whose content is also outlined in [15] and where this program has been developed.

In addition we impose three conditions. First, the following 'orthogonality' relations are supposed to hold:

$$\sum_{k\in I} w_k^2 w_n^2 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \begin{vmatrix} i & j & k \\ l & m & n' \end{vmatrix} = \delta_{n,n'} \,. \tag{2.3}$$

The summation is such that all symbols are defined, i.e. both 6-tuples (i,j,k,l,m,n) and (i,j,k,l,m,n') are admissible.

Secondly we assume that

$$w^2 = w_k^{-2} \sum_{i,j} w_i^2 w_j^2 \delta(i,j,k) \tag{2.4}$$

holds for all $k \in I$.

The following discussion relates $\tilde{w}^2$ and w^2. Define $c(l,k) = c(k,l) \in K$ by

$$c(l,k) = w_k^{-2} w_l^{-2} \sum_j \delta(l,j,k) w_j^2 \,. \tag{2.5}$$

Applying (2.3) twice gives

$$\begin{aligned} \sum_{i,j} w_i^2 w_j^2 \begin{vmatrix} i & j & k \\ l & m & k \end{vmatrix}^2 &= \sum_j \frac{w_j^2}{w_l^2} \delta(l,m,k)\delta(l,j,k) = w_k^2 c(l,k)\delta(l,m,k) \\ &= \sum_i \frac{w_i^2}{w_m^2} \delta(i,m,k)\delta(l,m,k) = w_k^2 c(m,k)\delta(l,m,k) \,. \end{aligned} \tag{2.6}$$

This gives

$$c(l,k) = c(m,k)$$

for all admissible triples (k,l,m).

Assume now I to be irreducible [16], i.e. to any $(i,j) \in I$ there exists a sequence $(l_1,\ldots,l_n) \in I$ with $l_1 = i, l_n = j$ and $\delta(l_\nu, l_{\nu+1}, l_{\nu+2}) = 1$ for all $(1 \leqslant \nu \leqslant n-2)$. An easy induction shows that $c(l,k) = c \in K$ for all $(k,l) \in I$ such that (2.5) takes the form

$$\sum_i w_i^2 \delta(i,j,k) = c w_j^2 w_k^2 \,. \tag{2.7}$$

Now (2.1), (2.4) and (2.7) combined give

$$w^2 = w_k^{-2} \sum_{i,j} w_i^2 w_j^2 \delta(i,j,k) = c\tilde{w}^2 \tag{2.8}$$

and hence $c \neq 0$ since $w \in K^*$ by assumption. Actually $c \in K^*$ since $c^{-1} = \tilde{w}^2 w^{-2}$. This in turn implies $\tilde{w}^2 = c^{-1} w^2$. Since $w_j^2 w_k^2 \in K^*$, the right-hand side of (2.7) is non-zero. This proves:

Lemma 2.1. Let I be irreducible and assume (2.3) and (2.4) to hold. Then to each $(i,j) \in I$ there exists a $k \in I$ such that (i,j,k) is admissible. Also $\tilde{w}^2 \in K^*$ and

$$\sum_i w_i^2 \delta(i,j,k) = \frac{w^2}{\tilde{w}^2} w_j^2 w_k^2 \tag{2.9}$$

holds for all $(j,k) \in I$.

Finally we remark that for the special case of quantum $6j$-symbols associated to $\mathrm{U}_q(\mathrm{sl}(2,\mathbf{C}))$ with $q = \exp(\mathrm{i}\pi s/r)$ (r and $s \in \mathbf{Z}$ relatively prime) one has

$$w_i = (\sqrt{-1})^{2i} \left(\frac{q^{2i+1} - q^{-2i-1}}{q - q^{-1}} \right)^{\frac{1}{2}} \qquad \left(i = 0, \frac{1}{2}, 1, \ldots, \frac{r}{2} - 1 \right)$$

and

$$w^2 = \tilde{w}^2 = \frac{-2r}{(q - q^{-1})^2} \tag{2.8$'$}$$

such that in particular $c = 1$.

From now on we will assume I to be irreducible. Finally we assume the following polynomial relation to hold in K (the Biedenharn–Elliot identity for $6j$-symbols):

$$\sum_n w_n^2 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \begin{vmatrix} i & n & m \\ D & A & C \end{vmatrix} \begin{vmatrix} j & l & n \\ D & C & B \end{vmatrix} = \begin{vmatrix} i & j & k \\ B & A & C \end{vmatrix} \begin{vmatrix} k & l & m \\ D & A & B \end{vmatrix} \tag{2.10$_2$}$$

again with restrictions similar to those in relation (2.3). We depict this graphically in figure 1 where the 6-tuple (i,j,k,l,m,n) is associated to the six edges of a tetrahedron and the elements $A, B, C, D \in I$ are associated to the vertices. Note that i and l belong to opposite edges and that in the triangle formed by the edges associated to i, j and k the vertex C is opposite to the edge k, etc.

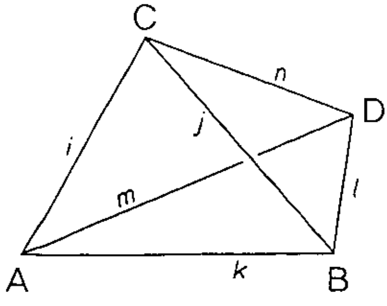

Figure 1. A tetrahedron with 6-tuple (i,j,k,l,m,n) and 4-tuple (A,B,C,D) associated to the edges and vertices, respectively, corresponding to (2.10_{0-4}).

Using (2.2), (2.3) and (2.4) one deduces from (2.10_2) four addition relations given as

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = w^{-2} \sum_{A,B,C,D} w_A^2 w_B^2 w_C^2 w_D^2 \begin{vmatrix} i & j & k \\ B & A & C \end{vmatrix} \cdot \begin{vmatrix} k & l & m \\ D & A & B \end{vmatrix} \begin{vmatrix} j & n & l \\ D & B & C \end{vmatrix} \begin{vmatrix} i & m & n \\ D & C & A \end{vmatrix} \tag{2.10$_0$}$$

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}\begin{vmatrix} i & n & m \\ D & A & C \end{vmatrix} = \sum_B w_B^2 \begin{vmatrix} i & j & k \\ B & A & C \end{vmatrix}\begin{vmatrix} k & l & m \\ D & A & B \end{vmatrix}\begin{vmatrix} j & n & l \\ D & B & C \end{vmatrix} \tag{2.10$_1$}$$

$$\sum_{\substack{l,m,n\\D}} w_l^2 w_m^2 w_n^2 w_D^2 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}\begin{vmatrix} i & n & m \\ D & A & C \end{vmatrix}$$

$$\cdot \begin{vmatrix} j & l & n \\ D & C & B \end{vmatrix}\begin{vmatrix} k & m & l \\ D & B & A \end{vmatrix} = w^2 \begin{vmatrix} i & j & k \\ B & A & C \end{vmatrix} \tag{2.10$_3$}$$

$$\sum_{\substack{i,j,k,l,m,n\\A,B,C,D}} w_i^2 w_j^2 w_k^2 w_l^2 w_m^2 w_n^2 w_A^2 w_B^2 w_C^2 w_D^2 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}\begin{vmatrix} i & n & m \\ D & A & C \end{vmatrix}$$

$$\cdot \begin{vmatrix} j & l & n \\ D & C & B \end{vmatrix}\begin{vmatrix} k & m & l \\ D & B & A \end{vmatrix}\begin{vmatrix} i & k & j \\ B & C & A \end{vmatrix} = (w^2)^3 \tilde{w}^2 . \tag{2.10$_4$}$$

Let M be a 3-manifold and X a triangulation of M which induces a triangulation ∂X of ∂M.

Definition 2.2. An edge colouring of X is a map $\underline{j} : \sigma^1 \mapsto j(\sigma^1)$ from the set of non-oriented 1-simplexes σ^1 of X into I. A vertex colouring of ∂X is a map $\underline{J} : \sigma^0 \mapsto J(\sigma^0)$ from the set of vertices of ∂X into I.

To a given edge colouring of X and to every non-oriented 3-simplex $\sigma^3 \in X$ we associate the $6j$-symbol

$$(6\underline{j})(\sigma^3) = \begin{vmatrix} j(\sigma_1^1) & j(\sigma_2^1) & j(\sigma_3^1) \\ j(\sigma_4^1) & j(\sigma_5^1) & j(\sigma_6^1) \end{vmatrix} \tag{2.11}$$

provided the 6-tuple $(j(\sigma_1^1), \ldots, j(\sigma_6^1))$ is admissible. Here σ_i^1 and σ_{i+3}^1 $(i = 1,2,3)$ are opposite edges in $\partial\sigma^3$.

To a given edge colouring of X, vertex colouring of ∂X and non-oriented 2-simplex $\sigma^2 \in \partial X$ we associate the $6j$-symbol

$$(6\underline{j},\underline{J})(\sigma^2) = \begin{vmatrix} j(\sigma_1^1) & j(\sigma_2^1) & j(\sigma_3^1) \\ J(\sigma_1^0) & J(\sigma_2^0) & J(\sigma_3^0) \end{vmatrix} \tag{2.12}$$

provided the 6-tuple $(j(\sigma_1^1), j(\sigma_2^1), j(\sigma_3^1), J(\sigma_1^0), J(\sigma_2^0), J(\sigma_3^0))$ is admissible. Here σ_i^0 are the vertices opposite to the edges $\sigma_i^1 (i = 1,2,3)$. Given an edge colouring of X and a vertex colouring of ∂X we define the 'Boltzmann–Gibbs' weight factor

$$W(X)(\underline{j},\underline{J}) = \prod_{\sigma^0 \in X} w^{-2} \prod_{\sigma^0 \in \partial X} w^2_{J(\sigma^0)} \prod_{\sigma^1 \in X} w^2_{j(\sigma^1)} \prod_{\sigma^3 \in X} (6\underline{j})(\sigma^3) \prod_{\sigma^2 \in \partial X} (6\underline{j},\underline{J})(\sigma^2) \tag{2.13}$$

provided again that all expressions on the right-hand side are defined.

For a given triangulation X of M as above we define the state sum to be given by

$$Z(X) = \sum_{\underline{j},\underline{J}} W(X)(\underline{j},\underline{J}) . \tag{2.14}$$

Here summation is over all such $\underline{j}$ and $\underline{J}$ for which the Boltzmann–Gibbs factor (2.13) is defined. For the special case $\partial M = \emptyset$, this agrees with the definition of Turaev and Viro.

Remark 2.3. A property of the state sum (2.14) is obvious: if M and hence X is not connected, the state sum is the product of the state sums for the connected components of X with the corresponding decomposition of ∂X. For example

$$Z(X_1 \amalg X_2) = Z(X_1) \cdot Z(X_2) \qquad \text{if} \quad X_1 \cap X_2 = \emptyset \,. \tag{2.15}$$

We now come to invariance under isotopies of the boundary ∂X of X. Basically this amounts to introducing a new triangulation of M and ∂M. By definition, an admissible elementary simple isotopy of ∂X consists of the addition (or subtraction) of a 3-simplex $\sigma^3 \in \hat{X}$ to X with the property that $\partial\sigma^3 \cap \partial X$ contains exactly r 2-simplexes ($1 \leqslant r \leqslant 3$) in X. Addition and subtraction are thus operations which are inverse to each other. In the case of our addition this changes $X_{\text{old}} = X$ to $X_{\text{new}} = X \cup \sigma^3$ and ∂X_{new} is obtained from $\partial X_{\text{old}} = \partial X$ by replacing the r 2-simplexes in $\partial\sigma^3 \cap \partial X_{\text{old}}$ by the remaining $(4-r)$ 2-simplexes in $\partial\sigma^3$.

Theorem 2.4. The state sum (2.14) is invariant under admissible, elementary simple isotopies of ∂X.

Proof. We only consider the addition of a 3-simplex since the argument for the inverse operation given by a subtraction is similar. We will relate these operations to the relations (2.10_{4-r}) $(r = 1, 2, 3)$ read from right to left. We now look at the individual terms in the state sum. Only if $r = 1$, X_{new} has an additional vertex compared to X_{old}. This is accounted for by the w^2 term in (2.10_3). To exhibit the other changes, we extend figure 1 by figure 2.

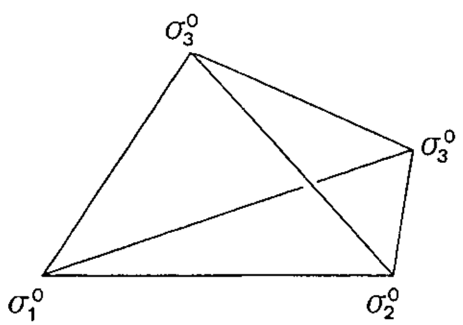

Figure 2. The tetrahedron $[\sigma_0^0, \sigma_1^0, \sigma_2^0, \sigma_3^0]$ with edge and vertex colourings as in figure 1.

Thus $j([\sigma_1^0, \sigma_3^0]) = i$, etc and $J(\sigma_1^0) = A$, etc. We then make the convention that for $r = 1$ the 2-simplex $[\sigma_1^0, \sigma_2^0, \sigma_3^0]$ is replaced by the remaining three 2-simplexes and similarly for $r = 2, 3$. From this the proof of the claim follows immediately if we make the choice

$$\underline{j}_{\text{new}} = \underline{j}_{\text{old}} \quad \text{and} \quad \underline{J}_{\text{new}} = \underline{J}_{\text{old}} \quad \text{on} \quad X_{\text{new}} \cap X_{\text{old}} \,. \tag{2.16}$$

So far we have not used relation (2.10_4). The geometric relevance of this relation is as follows. Consider the case where X is just a 3-simplex σ^3. This 3-simplex is removed using the rule (2.10_4) giving the following:

Theorem 2.5. With the notation and condition just employed the state sum satisfies the relation

$$Z(\sigma^3) = \frac{\tilde{w}^2}{w^2} = c^{-1}\,. \tag{2.17}$$

The converse result states what happens if one cuts out one 3-simplex σ^3. We apply the rule (2.10_0) to $(6\underline{j})(\sigma^3)$ and replace it by four $6j$-symbols associated to the four triangles on $\partial\sigma^3$, thus creating a 'hole' in X.

Theorem 2.6. The following relation is valid if σ^3 is contained in the interior of X:

$$Z(X \setminus \mathrm{int}\ \sigma^3) = w^2 Z(X)\,. \tag{2.18}$$

Theorems 2.4–2.6 represent the Stokes theorem referred to in the introduction.

The next step is to prove invariance under subdivisions. As in [16] we will resort to Alexander moves [1]. Actually any local subdivision will do. An Alexander move is defined as follows. Let σ^1 be an arbitrary 1-simplex in X and let $\mathtt{st}(\sigma^1)$ be its star. Let σ^0_* be the barycentre of σ^1. Outside $\mathtt{int}\ \mathtt{st}(\sigma^1) = \mathtt{st}(\sigma^1)\backslash\partial\mathtt{st}(\sigma^1)$ no new simplexes are introduced. In addition to σ^0_* the new simplexes in $\mathtt{int}\ \mathtt{st}(\sigma^1)$ are of the form $[\sigma^0_*, \sigma]$ where σ is any simplex in $\partial\,\mathtt{st}(\sigma^1)$.

By a well known theorem [1], two triangulations of M have a common subdivision under suitable iteration of Alexander moves on each of these two triangulations. We say that a triangulation X of M (inducing a triangulation ∂X of ∂M) is sufficiently fine if the following two conditions hold.

(a) the star of any 1-simplex not intersecting ∂X is homeomorphic to the closed unit ball in $\mathbf{R}^3$.

(b) For any 1-simplex σ^1 in X which intersects ∂X non-trivially, $\partial(X\backslash(\mathtt{int}\ \mathtt{st}(\sigma^1) \cup \mathtt{int}(\partial X \cap \partial\mathtt{st}(\sigma^1))))$ is simply isotopic to ∂X, i.e. may be obtained from ∂X by a succession of elementary simple isotopies.

Given any 3-manifold M there is obviously a sufficiently fine triangulation X of M. Also any subdivision of such a fine triangulation of M is again a fine triangulation.

Theorem 2.7. Given a sufficiently fine triangulation X of M, the state sum $Z(X)$ is invariant under Alexander moves.

Proof. Assume first that σ^1 intersects ∂X non-trivially. We use the invariance of theorem 2.4. By this theorem and conditions (a) and (b)

$$Z(X) = Z(\tilde{X}) \tag{2.19}$$

where $\tilde{X}$ is obtained from X by removing all simplexes in $\mathtt{int}\ \mathtt{st}(\sigma^1) \cup \mathtt{int}(\partial X \cap \partial\mathtt{st}(\sigma^1))$. In particular $\partial\tilde{X}$ is obtained from ∂X by replacing $\mathtt{int}(\partial X \cap \partial\mathtt{st}(\sigma^1))$ by its complement in $\partial\,\mathtt{st}(\sigma^1)$. We now perform the Alexander move associated to σ^1. This does not change the right-hand side of (2.19). Then we perform a deformation in the reverse order by adding all the new 3-simplexes in $\mathtt{int}\ \mathtt{st}(\sigma^1)$. Again this does not change the state sum. This concludes the proof for the case that σ^1 intersects ∂X non-trivially. Finally let $\sigma^1 \in X$ be such that it does not intersect ∂X. We now use only condition (a) coupled with relations (2.10_k) $(k = 0, 1, 2, 3)$ in the following way. Pick an arbitrary 3-simplex σ_0^3 in $\mathtt{st}(\sigma^1)$. We now apply theorem 2.6 to create a 'hole' in X and obtain

$$Z(X\backslash\mathtt{int}\ \sigma_0^3) = w^2 Z(X)\,. \tag{2.20}$$

Now let $\sigma_j^3 \in X$ $(0 \leqslant j \leqslant n)$ be the 3-simplexes in $\mathtt{st}(\sigma^1)$ such that σ_j^3 and $\sigma_{j+1}^3 (0 \leqslant j \leqslant n)$ with $\sigma_{n+1}^3 = \sigma_0^3$ have exactly one 2-simplex in common. We now inductively remove σ_j^3 from X $(1 \leqslant j \leqslant n-1)$ by applying the rule (2.10_1). Again this does not change the state sum (2.14). Finally we remove σ_n^3 by applying the rule (2.10_2). This also does not change the state sum (2.11). Thus we have arrived at a new space $X_{\mathrm{new}} = X\backslash\mathtt{int}\ \mathtt{st}(\sigma^1)$ with an additional boundary, i.e. $\partial X_{\mathrm{new}} = \partial X \cup \partial\mathtt{st}(\sigma^1)$. In other words, we apply our Stokes theorem and have

$$Z(X\backslash\mathtt{int}\ \mathtt{st}(\sigma^1)) = w^2 Z(X)\,. \tag{2.21}$$

We now perform the Alexander move associated to σ^1. This does not affect the left-hand side of (2.21). We now work with Stokes theorem in the opposite direction in the following way.

First observe that this Alexander move creates $2(n+1)$ 3-simplexes out of the original $(n+1)$ 3-simplexes of $\mathtt{st}(\sigma^1)$. More precisely, write σ^1 as $\sigma^1 = [\sigma_1^0, \sigma_2^0]$ and let σ_*^0 be the barycentre and set $\sigma_1^1 = [\sigma_1^0, \sigma_*^0]$, $\sigma_2^1 = [\sigma_*^0, \sigma_2^0]$. Let $\sigma_{j,i}^3$ $(0 \leqslant j \leqslant n, i = 1,2)$ be the new 3-simplexes such that $\sigma_{j,i}^3$ $(0 \leqslant j \leqslant n)$ form the star of σ_i^1 $(i = 1,2)$. Also the numeration is such that $\sigma_{j,1}^3$ and $\sigma_{j,2}^3$ result from the subdivision of σ_j^3. We now add $\sigma_{0,1}^3$ to $X\backslash\mathtt{int}\ \mathtt{st}(\sigma^1)$ by using the rule (2.10_3). This does not change (2.21). (Note that we have added the new vertex σ_*^0 during this process.) Then we iteratively add $\sigma_{j,1}^3$ $(1 \leqslant j \leqslant n-1)$ using rule (2.10_2) and then $\sigma_{n,1}^3$ by using rule (2.10_1). This again does not change (2.21). In the next step we add $\sigma_{0,2}^3$ by using rule (2.10_2) followed by adding successively $\sigma_{j,2}^3$ $(1 \leqslant j \leqslant n-1)$ with rule (2.10_1) and finally $\sigma_{n,2}^3$ using rule (2.10_0). This final step kills the extra w^2 factor appearing in (2.21). The invariance of the state sum under Alexander moves is thus completed.

The invariance of the state sum under subdivisions means that it does not depend on the specific triangulation X and hence defines an invariant of the 3-manifolds M:

$$Z(M) = Z(X)\,. \tag{2.22}$$

In particular we may use theorems 2.4–2.6 and write (2.17) and (2.18) as

$$Z(M \cup D^3) = c^{-1} Z(M) \tag{2.17$'$}$$

$$Z(M \setminus D^3) = w^2 Z(M) \tag{2.18$'$}$$

where a unit ball D^3 is added to M or cut out from M $(D^3 \subset M\backslash\partial M)$ respectively.

In the next section we shall explicitly compute the state sums for a selected set of examples.

3. Calculation of explicit examples

In this section we use two procedures, which may be called combinatorial surgery, to calculate the state sum $Z(M)$ for some examples. The first procedure ('handle cutting') applies to manifolds M which contain a handle

$$M = \tilde{M} \amalg (D^2 \times [0,1]) \qquad \text{such that} \quad \mathrm{genus}(\partial\tilde{M}) = \mathrm{genus}(\partial M) - 1\,. \tag{3.1}$$

The second type of surgery will be applied if

$$M = \tilde{M} \amalg (S^2 \times [0,1]) \qquad \text{with} \quad \partial\tilde{M} \cong \partial M \cup S^2 \cup S^2\,. \tag{3.2}$$

We also show that the state sum for closed, compact 3-manifolds satisfies a relation similar to the one obtained by Witten in his discussion of the Chern–Simons theory [17].

We choose M such that ∂M is a closed, compact, oriented, connected manifold of genus $g(\partial M)$ and Euler characteristics $\chi(\partial M) = 2 - 2g(\partial M)$ and such that after suitable iterations of (3.1) $\tilde{M} \cong D^3$ holds (i.e. M = solid surface = gefillte pretzel). With these assumptions and $c = w^2/\tilde{w}^2$ we have:

Theorem 3.1. The state sum is given by

$$Z(M) = c^{g(\partial M)-1}\,. \tag{3.3}$$

Remark 3.2. Using the factorization property (2.15) and theorem 2.6 this result generalizes to the case where ∂M may be not connected and M may contain N_{h} holes $\cong D^3$ (cf theorem 2.6)

$$Z(M) = w^{2N_{\mathrm{h}}} c^{-\chi(\partial M)/2+N_{\mathrm{h}}}\,. \tag{3.4}$$

This formula even holds for more general holes with boundary of genus $\neq 0$, if they are 'compressible', which means that relation (3.1) is fulfilled.

Remark 3.3. In the context of $\mathrm{U}_q(\mathrm{sl}(2,\mathbf{C}))$ we have $c = 1$ (see (2.8′)).

Proof. We first consider the case $M = D^3$ (the unit ball in $\mathbf{R}^3$) such that $\partial M \cong S^2$. Then by theorem 2.5 we have

$$Z(D^3) = \frac{\tilde{w}^2}{w^2} = c^{-1} \tag{3.5}$$

proving the claim for $g = 0$. We now proceed by induction of g, so let $g \geqslant 1$. The following discussion allows us to perform combinatorial surgery. Consider the manifold $D^2 \times [0,1]$ (D^2 the unit disc in $\mathbf{R}^2$) triangulated as in figure 3.

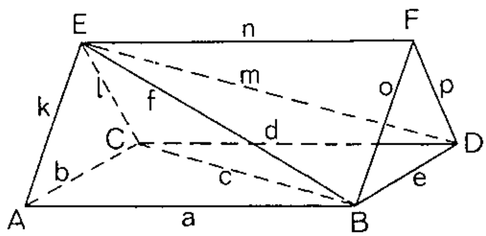

Figure 3. Triangulation of a piece of a handle (cf (3.1)).

We imagine this set $D^2 \times [0,1]$ to be cut out one of the handles of M such that $M \backslash (D^2 \times [0,1]) = \tilde{M}$ has a boundary $\partial \tilde{M}$ with genus $\tilde{g} = g - 1$. Hence we want to look at

$$W = \sum_{a,c,d,f,m,n} w_a^2 w_c^2 w_d^2 w_f^2 w_m^2 w_n^2 \begin{vmatrix} a & b & c \\ l & f & k \end{vmatrix} \begin{vmatrix} c & d & e \\ m & f & l \end{vmatrix} \begin{vmatrix} f & e & m \\ p & n & o \end{vmatrix} \begin{vmatrix} a & k & f \\ E & B & A \end{vmatrix} \cdot \begin{vmatrix} f & n & o \\ F & B & E \end{vmatrix} \begin{vmatrix} m & n & p \\ F & D & E \end{vmatrix} \begin{vmatrix} d & l & m \\ E & D & C \end{vmatrix} \begin{vmatrix} c & d & e \\ D & B & C \end{vmatrix} \begin{vmatrix} a & b & c \\ C & B & A \end{vmatrix} \tag{3.6}$$

i.e. the edge colourings k, l, b, e, p, o and the vertex colouring A, B, C, D, E, F are kept fixed in the following. We first perform the sum over a using (2.10_2) in the form

$$\sum_a w_a^2 \begin{vmatrix} a & b & c \\ e & f & k \end{vmatrix} \begin{vmatrix} a & k & f \\ E & B & A \end{vmatrix} \begin{vmatrix} a & b & c \\ C & B & A \end{vmatrix} = \begin{vmatrix} b & k & l \\ E & C & A \end{vmatrix} \begin{vmatrix} c & f & l \\ E & C & B \end{vmatrix} . \tag{3.7}$$

This means we use invariance under deformations of $\partial(D^2 \times [0,1])$ of theorem 2.3:

$$[E, B, A] \cup [C, B, A] \to [E, C, A] \cup [E, C, B] .$$

Similarly, we sum over n again using (2.10_2) in the form

$$\sum_n w_n^2 \begin{vmatrix} f & e & m \\ p & n & o \end{vmatrix} \begin{vmatrix} f & n & o \\ F & B & E \end{vmatrix} \begin{vmatrix} m & n & p \\ F & D & E \end{vmatrix} = \begin{vmatrix} e & f & m \\ E & D & B \end{vmatrix} \begin{vmatrix} e & o & p \\ F & D & B \end{vmatrix} \tag{3.8}$$

and summing analogously over c we find

$$W = \sum_{d,f,m} w_d^2 w_f^2 w_m^2 \begin{vmatrix} d & l & m \\ E & D & C \end{vmatrix}^2 \begin{vmatrix} e & f & m \\ E & C & A \end{vmatrix}^2 \begin{vmatrix} b & k & l \\ E & C & A \end{vmatrix} \begin{vmatrix} l & o & p \\ F & D & B \end{vmatrix} . \tag{3.9}$$

We now perform the sum over d and m using (2.3) and (2.7) to obtain

$$\begin{aligned} W &= \sum_f w_E^{-2} w_B^{-2} w_f^2 \begin{vmatrix} b & k & e \\ E & C & A \end{vmatrix} \begin{vmatrix} e & o & p \\ F & D & B \end{vmatrix} \delta(E, B, f) \\ &= c \begin{vmatrix} b & k & l \\ E & C & A \end{vmatrix} \begin{vmatrix} e & o & p \\ F & D & B \end{vmatrix} . \end{aligned} \tag{3.10}$$

Comparing this with $Z(\tilde{M})$ obtained from a triangulation induced by that of M, we have

$$Z(M) = c \; Z(\tilde{M}) . \tag{3.11}$$

This concludes the induction and the proof of theorem 3.1 is completed.

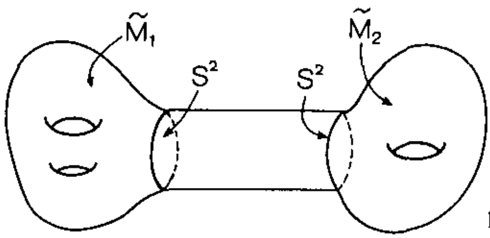

Figure 4. The two-dimensional analogue of (3.12).

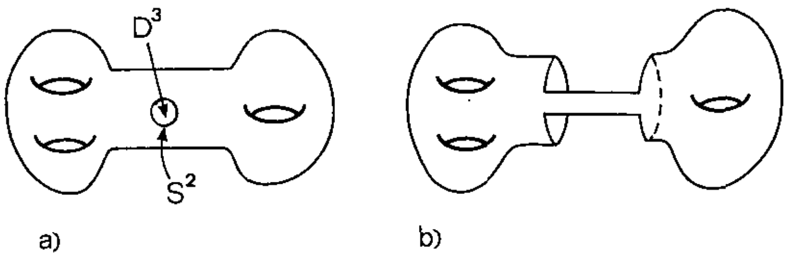

Figure 5. The two-dimensional analogue of (*a*) cutting a hole (cf (3.13)) and (*b*) deforming $\partial\tilde{M}$.

In the next step we mimic the surgery considered in [17]. As a special case of (3.2) for disconnected $\tilde{M}_1$ and $\tilde{M}_2$ let

$$M = \tilde{M}_1 \underset{S^2}{\cup} (S^2 \times [0,1]) \underset{S^2}{\cup} \tilde{M}_2 \qquad \tilde{M}_1 \cap \tilde{M}_2 = \emptyset \tag{3.12}$$

(figure 4 gives the two-dimensional analogue).

We apply the relations (2.10_k) $(0 \leqslant k \leqslant 3)$ in the set $S^2 \times [0,1]$. Using theorem 2.6 we introduce a hole D^3 (the unit ball in $\mathbf{R}^3$) with boundary S^2 (cf figure 5(*a*) as a two-dimensional analogue)

$$Z(M) = w^{-2} Z(\tilde{M}) \qquad \text{with} \quad \tilde{M} = M \setminus D^3 . \tag{3.13}$$

Then we deform the hole using the invariance of theorem 2.4 according to figure 5(*b*), such that $\tilde{M}_1$ and $\tilde{M}_2$ are connected by a tube of that type depicted in figure 3. Cutting this tube we obtain from (3.11) and (3.13) (cf figure 6(*a*))

$$Z(M) = w^{-2} c\ Z(\tilde{M}_1) \cdot Z(\tilde{M}_2) . \tag{3.14}$$

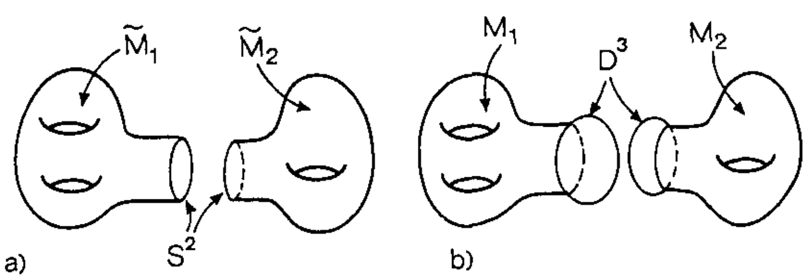

Figure 6. The two-dimensional analogue of (*a*) cutting the manifold $\tilde{M}$ and (*b*) gluing back two 3-balls to the boundaries.

To proceed further, we glue D^3 (viewed as a 3-simplex) back to $\tilde{M}_1$ and $\tilde{M}_2$, respectively, along the common boundary $\cong S^2$ to obtain $M_i = \tilde{M}_i \underset{S^2}{\cup} D^3$ $(i = 1, 2)$ (again figure 6(*b*) describes the two-dimensional analogue). By theorem 2.6 we have

$$Z(\tilde{M}_i) = w^2 Z(M_i) \qquad (i = 1, 2). \tag{3.15}$$

From (3.13)–(3.15) we conclude

$$Z(M) = w^2 c \cdot Z(M_1) \cdot Z(M_2). \tag{3.16}$$

Therefore we have the the following theorem, which compares with relation (4.2) in [17].

Theorem 3.4. With the above notations, the following relations are valid:

$$\frac{Z(M)}{Z(S^3)} = \frac{Z(M_1)}{Z(S^3)} \frac{Z(M_2)}{Z(S^3)} \tag{3.17}$$

$$Z(S^3) = \frac{1}{cw^2} \tag{3.18}$$

$$Z(S^2 \times S^1) = 1. \tag{3.19}$$

Proof. Note that for $M = S^3$ we also have $M_1 \cong M_2 \cong S^3$ (cf figures 5 and 6 for the two-dimensional analogue). Therefore (3.16) implies (3.18) and finally (3.17). To prove (3.19), which compares with relation (4.31) in [17], we consider equation (3.2) for $M = S^2 \times S^1$ which means that also $\tilde{M} \cong S^2 \times [0, 1]$. In analogy to relation (3.16) we have

$$Z(S^2 \times S^1) = w^2 c\ Z(D^3 \underset{S^2}{\cup} \tilde{M} \underset{S^2}{\cup} D^3). \tag{3.20}$$

However, with $D^3 \underset{S^2}{\cup} \tilde{M} \underset{S^2}{\cup} D^3 \cong S^3$ and (3.18) we find (3.19).

Remark 3.5. In the context of $\mathrm{U}_q(\mathrm{sl}(2, \mathrm{C}))$ we have with $q = \exp(i\pi s/r)$ (r and s relatively prime) (see (2.8′))

$$w^{-2} = \tilde{w}^{-2} = \frac{2}{r} \sin^2 \frac{\pi s}{r}. \tag{3.21}$$

This compares with relation (2.26) in [17] for the choice $r = k + 2$ and $s = 1$. In other words, the state sum of Turaev and Viro should be compared with the (absolute) square of the state sum of Witten for the Chern–Simons theory. Note that our state sum is independent of any orientation, while the definition for the Chern–Simon theory depends of a choice of the orientation. In fact there is a rigorous definition [12] $\tau_q(M)$ for the state sum of Witten's theory such that for oriented M $\tau_q(M)\tau_q(-M) = |\tau_q(M)|^2 = Z(M)$ [14].

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