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FINITE-SIZE CORRECTIONS FOR INTEGRABLE SYSTEMS AND CONFORMAL PROPERTIES OF SIX-VERTEX MODELS

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Statistical systems at second-order phase-transition points should exhibit conformal invariance at long distances. Their conformal properties can be analysed by investigating finite-size scaling behaviour. For integrable lattice models in two dimensions, methods are proposed to calculate, from the Bethe ansatz solution, the conformal anomaly c and all scaling dimensions. As an application results for the q-state Potts model and modified six-vertex models are presented.

1. Introduction

This work continues a previous one [1] written in collaboration with H. de Vega. Using the Bethe ansatz solution we calculated the central charges of the Virasoro algebra for the critical six-vertex and q-state Potts models. These results were obtained analysing the finite size behaviour of ground state energies. In addition a first attempt was made to consider for the six-vertex model the lowest excitation and we calculated the scaling dimension of the corresponding field. The method used there was based on procedures developed for noncritical models [3] (see also refs. [4, 26]). In the present work* this method is extended in a non-trivial way to arbitrary excitations of the six-vertex and q-state Potts models. Similar methods have been applied to Heisenberg models in refs. [5, 6].

Conformal invariance of two-dimensional quantum field theories has become a subject of main interest because of its connection to string theory. Statistical mechanical systems at second-order phase-transition points (i.e. critical systems) should possess conformal invariance [7]. We shall use this fact to obtain information on conformal quantum field theories. It would be interesting to find *integrable* lattice models which yield these conformal quantum field theories in the scaling limit. Since a direct calculation of the central charge of the Virasoro algebra and the conformal dimensions for specific models is usually not simple, it seems worthwhile to determine them for integrable models from the Bethe ansatz solution.

* Some results have been published previously without proofs [2].

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This will be done for the six-vertex model (the critical eight-vertex model) and for some modified six-vertex models related to Potts models.

In two dimensions, conformal invariance is a rather strong restriction since the transformation group is infinite dimensional, in contrast to higher dimensions. It is related to the Virasoro algebra via [8]

$$[L_n, L_m] = (n-m) L_{n+m} + \tfrac{1}{12} c (n^3 - n)\, \delta_{n,-m},$$

$$c = \text{“central charge”}, \tag{1.1}$$

for which a well-developed theory exists [9]. Belavin, Polyakov and Zamolodchikov [10] classified conformal quantum field theories in terms of the central charge

$$c = 1 - 6/\nu(\nu-1), \tag{1.2}$$

where ν is a rational number if there exists only a finite number of "primary fields", i.e. fields transforming under conformal transformations

$$z = x + iy \to w(z),$$

$$\bar{z} = x - iy \to \bar{w}(\bar{z}), \tag{1.3}$$

like

$$A(z, \bar{z}) \to \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{\Delta} \left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right)^{\bar{\Delta}} A(w, \bar{w}). \tag{1.4}$$

The conformal dimensions can assume values given by the Kac formula [9]

$$\Delta, \bar{\Delta} = h_{pq} = \frac{(\nu p - (\nu-1) q)^2 - 1}{4\nu(\nu-1)}, \qquad 1 \leqslant q \leqslant p < \nu - 1,\ p, q = \text{integer}. \tag{1.5}$$

Friedan et al. [14] have shown that unitarity restricts the number ν to

$$\nu = \text{integer} \geqslant 2 \qquad \text{or} \qquad c \geqslant 1. \tag{1.6}$$

It would be interesting to find integrable lattice models which yield, in the scaling limit, conformal quantum field theories corresponding to these unitary representations of the Virasoro algebra. Therefore, it is useful to calculate the central charge and the conformal dimensions for integrable critical statistical models. This can be done by analysing finite-size effects.

For a statistical system on a periodic $N \times M$ square lattice one introduces a transfer matrix τ. The partition function can be written as

$$Z = \operatorname{tr} \tau^M = \sum_i \lambda_i^M, \tag{1.7}$$

where the sum extends over all eigenvalues of the transfer matrix. Cardy [12] has shown that conformal invariance implies for the maximal eigenvalue

$$\lambda_{\max} \approx \exp\left(-Nf_\infty + \frac{1}{N}\frac{\pi}{6}c \right), \qquad \text{for} \qquad N \to \infty, \tag{1.8}$$

where f_∞ is the free energy per site (in the thermodynamic limit) and c is the central charge of the Virasoro algebra (cf. eq. (1.1)).

Introducing in addition the "crossed" transfer matrix $\hat{\tau}$ (defined on an M-chain) we obtain for $1 \ll N \ll M$ two expressions for the partition function, i.e.

$$Z \approx \exp(-NMf) \sum_i \left(\hat{\lambda}_i / \hat{\lambda}_{\max} \right)^N, \tag{1.9a}$$

$$Z \approx \exp\left(-NMf + \frac{M}{N}\frac{\pi}{6}c \right), \tag{1.9b}$$

where only the "low-energy excitations" with $\hat{\lambda}_i \approx \hat{\lambda}_{\max}$ contribute to the sum in eq. (1.9a). Hence, two methods to investigate the partition function on a strip $1 \ll N \ll M \to \infty$ are available:

(a) one considers the transfer matrix $\hat{\tau}$ for an infinite chain ($M \to \infty$) and calculates the large-N corrections to the sum in eq. (1.9a) or;

(b) one considers the transfer matrix τ on an N-sites chain and calculates the large-N corrections to the maximal eigenvalue $\lambda_{\max}$ according to eq. (1.9b). By both of these procedures one obtains the central charge of the model.

In addition, the conformal dimensions of operators of the model can be determined if one looks for excited states with energy $E = -\mathscr{R} \log \lambda$ and momentum $P = -\mathscr{I} \log \lambda$. As is argued in ref. [12] one has for $N \to \infty$

$$E_i - E_0 \approx 2\pi d_i / N,$$

$$P_i - P_0 \approx 2\pi s_i / N, \tag{1.10}$$

where $d_i = \Delta + \bar{\Delta}$ ($s_i = \Delta - \bar{\Delta}$) is the "scaling dimension" ("spin") of the operator A associated to the excitation ψ_i, i.e. $\langle 0 | A | \psi_i \rangle \neq 0$.

A direct calculation of the central charge and the conformal dimensions for specific models is usually not simple. Therefore, it seems worthwhile to determine them for integrable models from the Bethe ansatz solution. This will be done in the following for the six-vertex model and for some modified six-vertex models related to Potts models and other models of interest. Method (a) has been developed in sects. 3 and 4 of ref. [1], method (b) has been applied in sect. 4 of ref. [1] and will be extended in a non-trivial way in the following.

2. Six-vertex model – Bethe ansatz

The six-vertex model is a classical statistical lattice model in two dimensions. Its partition function on an $N \times M$ periodic square lattice L is

$$Z = \sum_{\text{conf.}} \prod_{x \in L} \omega(x), \tag{2.1}$$

where the sum extends over all allowed "bond configurations". Each bond can accept one of two states characterized by arrows. The configuration is allowed if at each vertex x there are two incoming and two outgoing arrows. The vertex weights $\omega(x)$ are determined by the six bond configurations at the lattice site x depicted in fig. 1. They assume the following values [13, 14]

$$\begin{aligned} \omega_1 &= \omega_2 = a = \sin(\gamma - \theta), \\ \omega_3 &= \omega_4 = b = \sin\theta, \\ \omega_5 &= \omega_6 = c = \sin\gamma. \end{aligned} \tag{2.2}$$

The parametrization in terms of the "spectral parameter" θ and the "coupling" γ has been introduced for later convenience. A trivial overall factor is set equal to one.

The partition function with periodic boundary conditions

$$Z = \operatorname{tr}(\tau(\theta, \gamma))^M \tag{2.3}$$

can be written in terms of a *transfer matrix* $\tau(\theta, \gamma)$ defined on a periodic chain of length N. The eigenstates and eigenvalues of the transfer matrix can be calculated

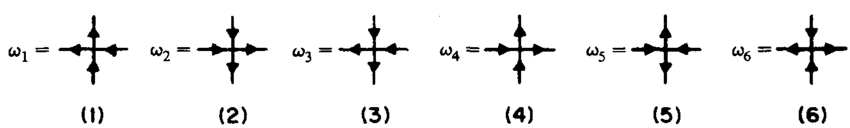

Fig. 1. Allowed vertex configurations.

by solving the "Bethe ansatz equations" (see, e.g. refs. [1,13,14])

$$\left(\frac{\sinh\frac{1}{2}(q_j - i\gamma)}{\sinh\frac{1}{2}(q_j + i\gamma)}\right)^N \prod_{i=1}^{n} \frac{\sinh\frac{1}{2}(q_j - q_i + 2i\gamma)}{\sinh\frac{1}{2}(q_j - q_i - 2i\gamma)} e^{i2\kappa\gamma} = -1,$$

$$j = 1,\dots,n, \qquad 0 \leqslant n \leqslant \tfrac{1}{2}N. \tag{2.4}$$

The eigenvalues $\lambda(\theta,\gamma)$ of the transfer matrix $\tau(\theta,\gamma)$ can be expressed in terms of the roots of these equations

$$\lambda = \lambda_A + \lambda_D, \qquad \lambda_D(\theta,\gamma) = \lambda_A^*(\gamma - \theta^*, \gamma),$$

$$\lambda_A(\theta,\gamma) = a^N \prod_{i=1}^{n} (-1) \frac{\sinh\frac{1}{2}(q_i + 2i\theta + i\gamma)}{\sinh\frac{1}{2}(q_i + 2i\theta - i\gamma)} e^{-i\kappa\gamma}. \tag{2.5}$$

Since $|\lambda_D| < |\lambda_A|$ holds for $0 < \theta < \frac{1}{2}\gamma$ only λ_A contributes to the partition function for large M. The constant phases $2\kappa\gamma$ in eq. (2.4) and $-\kappa\gamma$ in eq. (2.5) have been introduced for later convenience. For the ordinary six-vertex model, $\kappa = 0$ holds. For other models, e.g. Potts models, we shall have $\kappa \neq 0$.

There exist detailed investigations of the Bethe ansatz equations (2.4) (see e.g. ref. [15]). The analysis simplifies for the discrete values of the "coupling"

$$\gamma = \pi/\nu, \qquad \nu = 2,3,4,\dots. \tag{2.6}$$

We are interested in these values. So we assume the validity of eq. (2.6) in the following. For large N the Bethe ansatz equations have solutions of "string" type

(i) k^+string

$$q = q^{(k)} + i\gamma(-k+1, -k+3, \dots, k-1), \qquad k = 1,\dots,\nu-1, \tag{2.7}$$

(ii) 1^-string

$$q = q^{(a)} + i\pi. \tag{2.8}$$

The k^+ and 1^- strings may be interpreted as "bound states" of k pseudoparticles and as an "antipseudoparticle", respectively. For simplicity, I will not consider antipseudoparticles in the following. (We shall see later that they do not contribute to the minimal energy eigenstates for the Potts model.) The Bethe ansatz equations can be written in terms of the real "rapidities" $q^{(k)}$ of these states as

$$\exp\left(iNp_k\left(q_j^{(k)}\right)\right) \prod_{l=1}^{\nu-1} \prod_{i=1}^{n_l} \exp\left(\Phi_{kl}\left(q_j^{(k)} - q_i^{(l)}\right)\right) e^{i2k\kappa\gamma} = -1, \tag{2.9}$$

where the "momenta" p_k and the "scattering phase shifts" Φ_{kl} are obtained by taking products of the corresponding terms in eq. (2.4) (see e.g. ref. [1]). The Fourier transforms $\tilde{f}(x)=\int dq/(2\pi)e^{iqx}f(q)$ of their derivatives are

$$\tilde{p}'_k(x)=\frac{\sinh(\pi-k\gamma)x}{\sinh\pi x}, \tag{2.10}$$

$$\tilde{\Phi}'_{kl}(x)=\delta_{kl}-\frac{2\cosh\gamma x\sinh(\pi-l\gamma)x\sinh k\gamma x}{\sinh\pi x\sinh\gamma x}, \qquad 1\leqslant k\leqslant l\leqslant\nu-1. \tag{2.11}$$

Note the identity which will be used later

$$\delta_{1k}-\tilde{\Phi}'_{1k}(x)=2\cosh\gamma x\tilde{p}'_k(x). \tag{2.12}$$

We introduce the phase functions appearing in eq. (2.9) (cf. ref. [3])

$$z_k(q)=Np_k(q)+\sum_{l,i}\Phi_{kl}\left(q-q_i^{(l)}\right)+2k\kappa\gamma. \tag{2.13}$$

Then, the logarithm of the Bethe ansatz equation (2.9) yields

$$2\pi I_j^{(k)}=z_k\left(q_j^{(k)}\right)\begin{cases} I_j^{(k)}\in\left(\boldsymbol{Z}+\frac{1}{2}\right)\cap 1/2\pi\,]z_k(-\infty),z_k(\infty)[\\ j=1,\dots,n_k.\end{cases} \tag{2.14}$$

The asymptotic values of the phase functions are

$$\begin{aligned} z_k(-\infty)&=2\pi\sum_l\left(2\min(k,l)-\delta_{kl}\right)n_l+2k\gamma\Sigma+2k\kappa\gamma,\\ z_k(\infty)&=2\pi N-2k\gamma\Sigma+2k\kappa\gamma, \end{aligned} \tag{2.15}$$

where $\Sigma=\frac{1}{2}N-n$ is the "spin" of the Bethe state and n_l is the number of l-strings. We observe that sets of roots $\{q_1^{(k)},\dots,q_{n_k}^{(k)}\}$ are given by sets of numbers $\{I_1^{(k)},\dots,I_{n_k}^{(k)}\}$. For the "ground state" (corresponding to the maximal eigenvalue λ) there are only real roots ($k=1$) without holes, which means that the numbers $I_j^{(1)}$ accept all possible values in an interval $[I_{\min}^{(1)},I_{\max}^{(1)}]$. For "excited states" there may be "holes" in the distribution of the $I_j^{(1)}$ and higher strings ($k>1$) may appear.

In order to solve the Bethe ansatz equations we denote the derivatives of the phase functions by

$$\sigma_k(q)=\frac{d}{dq}z_k(q), \tag{2.16}$$

introduce the densities of k-strings

$$\rho_k(q) = 2\pi \sum_{i=1}^{n_k} \delta\left(q - q_i^{(k)}\right), \tag{2.17}$$

and the "density of holes"

$$\varphi(q) = \sigma(q) - \rho(q), \qquad \sigma = \sigma_1 . \tag{2.18}$$

The latter name is motivated by the fact that

$$\int_Q^{Q'} \frac{\mathrm{d}q}{2\pi} \varphi(q) = I^{(1)}(Q') - I^{(1)}(Q) - \#\, \text{roots} \in [Q, Q']$$

is the number of possible minus the number of actual real roots in an interval $[Q, Q']$. But note that only in the limit N, $n \to \infty$, where the roots become dense on the real line, is the function φ of the form

$$\varphi(q) = 2\pi \sum_{\text{holes}} \delta(q - q_{\text{h}}) . \tag{2.19}$$

Taking the derivative $\mathrm{d}/\mathrm{d}q$ of eq. (2.13) we obtain with eqs. (2.16)

$$\sigma_k = N p_k' + \sum_l \Phi_{kl}' * \rho_l , \tag{2.20}$$

where the convolution is defined by $(f * g)(q) = \int \mathrm{d}q'/(2\pi) f(q - q') g(q')$. Since we are looking for the ground state and low-energy excitations which are characterized by a large number of real roots (1^+ strings), a few holes and complex roots (k-strings, $k > 1$) we use eq. (2.18) and write eq. (2.20) as follows

$$\sigma_k = \rho_0 \delta_{k1} - J_k * \varphi + \sum_{l>1} J_{kl} * \rho_l \tag{2.21}$$

where

$$\rho_0 = N(1 - \Phi_{11}')^{-1} * p_1' ,$$

$$J_k = (1 - \Phi_{11}')^{-1} * \Phi_{k1}' ,$$

$$J_{kl} = \Phi_{k1}' * J_l + \Phi_{kl}' . \tag{2.22}$$

In deriving eq. (2.21) the identity eq. (2.12) has been used. The first term on the r.h.s. of eq. (2.21) represents the root density for the ground state in the thermodynamic limit $N \to \infty$. It can be calculated by means of eqs. (2.10) and (2.11)

$$\rho_0(q) = N \frac{\frac{1}{2}\nu}{\cosh \frac{1}{2}\nu q}, \qquad \gamma = \frac{\pi}{\nu}. \tag{2.23}$$

The second term in eq. (2.21) correspond to holes in the distribution of real roots and the third term to complex roots. The second term also contains the finite-size corrections which will be investigated in sect. 3.

The eigenvalues of the transfer matrix are obtained from eqs. (2.5), (2.17), (2.18) and (2.22)

$$\begin{aligned} \log \lambda_A(\theta, \gamma) &= -\sum_k \int \frac{\mathrm{d}q}{2\pi} i\big(p_k(q+2i\theta) - k\pi\big)\rho_k(q) - i\kappa\gamma \\ &= -Nf_\infty + \int \frac{\mathrm{d}q}{2\pi} \varepsilon(q)\varphi(q) + i\pi \sum_k n_k(k-1) - i\kappa\gamma, \end{aligned} \tag{2.24}$$

where

$$\begin{aligned} \varepsilon(q) &= i\Big(\big(1 - \Phi'_{11}\big)^{-1} * \big(p_1(\cdot + 2i\theta) - \pi\big)\Big)(q) \\ &= i \sin^{-1} \tanh\big(\tfrac{1}{2}\nu q + 2i\theta\big). \end{aligned} \tag{2.25}$$

The first term in eq. (2.24) gives the free energy per site in the thermodynamic limit

$$f_\infty = -\log a - \frac{1}{N} \int \frac{\mathrm{d}q}{2\pi} i\big(p_1(q+2i\theta) - \pi\big)\rho_0(q).$$

The second term corresponds to holes and the finite-size corrections. From eqs. (2.12) and (2.21) it follows that the third term corresponding to the complex roots contributes only a constant phase.

3. Finite-size corrections

We are interested in the ground state and low-energy excitations with a finite number of holes and complex roots "near to $q = \pm\infty$". Hence, we consider an interval $[q_-, q_+]$ which does not contain any holes such that $q_\pm \to \pm\infty$ for $N \to \infty$. In order to discuss finite-size behaviour we have to investigate the second terms in eqs. (2.21) and (2.24) containing the function

$$\varphi(q) = \sigma(q) - 2\pi \sum_{i=1}^{n_1} \delta(q - q_i),$$

defined by eq. (2.18). Let us first consider eq. (2.24). We have to analyse

$$\int_{-\infty}^{+\infty}\frac{\mathrm{d}q}{2\pi}\varepsilon(q)\varphi(q)=\int_{q_-}^{q_+}\frac{\mathrm{d}q}{2\pi}\varepsilon(q)\varphi(q)$$

$$+\left(\int_{q_+}^{+\infty}+\int_{-\infty}^{q_-}\right)\frac{\mathrm{d}q}{2\pi}\varepsilon(q)\varphi(q)-\sum_i{}'\varepsilon(q_i)\tag{3.1}$$

where Σ' means that the sum extends only over those real roots q_i which are not in $[q_-,q_+]$.

After the substitution

$$I(q)=\frac{1}{2\pi}z_1(q),\qquad I_\pm=I(q_\pm),\tag{3.2}$$

where $z_1(q)$ is defined by eq. (2.13), we can apply the Euler–Maclaurin formula (A.5) derived in appendix A (cf. ref. [5]). The first term in eq. (3.1) then reads

$$-\tfrac{1}{2}(\varepsilon(q_+)+\varepsilon(q_-))-\tfrac{1}{6}\pi\left(\frac{\varepsilon'(q_+)}{\sigma(q_+)}-\frac{\varepsilon'(q_-)}{\sigma(q_-)}\right)+\Delta_1,\tag{3.3}$$

where

$$\Delta_1=-\int_{I_-}^{I_+}\mathrm{d}I(\mathrm{d}/\mathrm{d}I)^3\varepsilon(q(I))g_3(I),\tag{3.4}$$

and $g_3(I)$ is a periodic function with period 1 defined by eq. (A.4). Using the asymptotic expansion of $\varepsilon(q)$

$$\varepsilon(q)=\tfrac{1}{2}i\pi-2i\,\mathrm{e}^{-\nu(q+i2\theta)/2}+\mathrm{O}(\mathrm{e}^{-3\nu q/2}),\quad\text{for}\quad q\to\infty,\tag{3.5}$$

we obtain for the second term in eq. (3.1)

$$\tfrac{1}{2}i\pi\tilde{\tau}_+(0)-2i\,\mathrm{e}^{-i\nu\theta}\mathrm{e}^{-\nu q/2}+\tilde{\tau}_+\left(\tfrac{1}{2}i\nu\right)+\mathrm{O}\left(\tilde{\tau}_+\left(\tfrac{3}{2}i\nu\right)\mathrm{e}^{-3\nu q_+/2}\right),\tag{3.6}$$

where the following notation has been used

$$\tilde{\tau}_+(x)=\int_0^\infty\frac{\mathrm{d}q}{2\pi}\,\mathrm{e}^{iqx}\tau_+(q),\qquad\tau_+(q)=\sigma(q+q_+).\tag{3.7}$$

A similar expression to eq. (3.5) where q_+ is replaced by q_- is obtained for the third term in eq. (3.1).

Inserting eqs. (3.3) and (3.6) in the finite-size correction term of the eigenvalue given by eq. (3.1) we obtain

$$\int_{-\infty}^{+\infty}\frac{\mathrm{d}q}{2\pi}\varepsilon(q)\varphi(q)=\tfrac{1}{2}i\pi(\tilde{\tau}_+(0)-\tilde{\tau}_-(0))+\delta_1+\delta_2+\delta_3+\Delta_1$$

$$+i\,\mathrm{e}^{-i\nu\theta}\mathrm{e}^{-\nu q_+/2}\left(1-\tfrac{1}{6}\pi\frac{\nu}{\sigma(q_+)}-2\tilde{\tau}_+\left(\tfrac{1}{2}i\nu\right)\right)$$

$$-i\,\mathrm{e}^{i\nu\theta}\mathrm{e}^{\nu q_-/2}\left(1-\tfrac{1}{6}\pi\frac{\nu}{\sigma(q_-)}-2\tilde{\tau}_-\left(\tfrac{1}{2}i\nu\right)\right)-\sum_i{}'\varepsilon(q_i), \quad (3.8)$$

where

$$\delta_1=\mathrm{O}(\mathrm{e}^{-3\nu|q_\pm|/2}),\qquad \delta_2=\mathrm{O}\left(\tilde{\tau}_\pm\left(\tfrac{3}{2}i\nu\right)\delta_1\right),\qquad \delta_3=\mathrm{O}\left(\frac{\delta_1}{\sigma(q_\pm)}\right). \quad (3.9)$$

We will later see that the δ's can be neglected for large N whereas Δ_1 gives a finite contribution which is, however, canceled by similar terms in $\tilde{\tau}_\pm(\frac{1}{2}i\nu)$.

In order to calculate $\tilde{\tau}_\pm(\frac{1}{2}i\nu)$ we apply analogous techniques to those above to eq. (2.21) which for $k=1$ determines $\sigma(q)$ as

$$\sigma=\rho_0-J*\varphi+\sum_{k>1}J_k*\rho_l,\qquad J=J_1. \quad (3.10)$$

We first concentrate on this function at $q>q_+$ to determine τ_+ (cf. eq. (3.7)). The decomposition of eq. (3.10), analogously to eq. (3.1), leads after the shift $q\to q+q_+$ to

$$\tau_+(q)=f(q)-\int_0^\infty\frac{\mathrm{d}q'}{2\pi}J(q-q')\tau_+(q'), \quad (3.11)$$

where the function $f(q)$ is obtained in the same way as eqs. (3.3)

$$f(q)=\rho_0(q+q_+)+\tfrac{1}{2}J(q)-\tfrac{1}{6}\pi\frac{1}{\sigma(q_+)}J'(q)+\sum_{k,i}{}'J_k\left(q+q_+-q_i^{(k)}\right)$$

$$+\tfrac{1}{2}J(q+q_+-q_-)-\tfrac{1}{6}\pi\frac{1}{\sigma(q_-)}J'(q+q_+-q_-)$$

$$-\int_{-\infty}^{q_-}\frac{\mathrm{d}q'}{2\pi}J(q+q_+-q')\sigma(q')+\Delta_2$$

$$=f_0(q)+f_1(q)+\delta_4+\delta_5+\delta_6+\Delta_2. \quad (3.12)$$

The sign Σ' means that for $k=1$ the sum extends only over those real roots q_i which are not in $[q_-, q_+]$. The functions f_0 represent the asymptotic contributions from the first three terms and f_1 those from the complex and real roots outside $[q_-, q_+]$

$$f_0(q) = N\nu\, e^{-\nu q/2} + e^{-\nu q/2} + \tfrac{1}{2}J(q) - \tfrac{1}{6}\pi \frac{1}{\sigma(q_+)} J'(q),$$

$$f_1(q) = \sum_{k,i}{}' J_k\left(q + q_+ - q_i^{(k)}\right). \tag{3.13}$$

The small terms to be neglected are

$$\delta_4 = \mathrm{O}\left(e^{-3\nu q/2}\delta_1\right), \qquad \delta_5 = \mathrm{O}\left(e^{-\nu(q+q_+-q_-)/(\nu-1)}\right), \qquad \delta_6 = \mathrm{O}\left(\delta_5/\sigma(q_-)\right), \tag{3.14}$$

where the asymptotic behaviour $J(q) = \mathrm{O}(\exp(-\nu q/(\nu-1)))$ has been used. The term

$$\Delta_2 = -\int_{I_-}^{I_+} \mathrm{d}I (\mathrm{d}/\mathrm{d}I)^3 J(q - q(I)) g_3(I), \tag{3.15}$$

gives a finite contribution which is, however, canceled in the final result by other similar terms. Eq. (3.11) for the function $\tau_+(q)$ is an integral equation of Wiener–Hopf type similar to that considered by Yang and Yang [16]. The solution (see appendix B) is given by eq. (3.7) and

$$\tilde{\tau}_+(x) = G^+(x)\frac{1}{2\pi i}\int \frac{\mathrm{d}y}{y - x - i\varepsilon} G^-(y)\tilde{f}(y), \tag{3.16}$$

where

$$\tilde{f}(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{2\pi}\, e^{iqx} f(q). \tag{3.17}$$

The functions $G^{\pm}(x)$ are uniquely defined by (cf. eqs. (2.11) and (2.22))

$$G^+(x)G^-(x) = \frac{1}{1+\tilde{J}(x)} = \frac{2\cosh\gamma x \sinh(\pi-\gamma)x}{\sinh \pi x},$$

$$G^-(x) = G^+(-x), \qquad G^{\pm}(\infty), \tag{3.18}$$

and the conditions that $G^+(x)$ is analytic and different from zero in the upper half plane $\mathscr{I}x \geqslant 0$.

In addition to the integral equation (3.11) there are further restrictions of $\tau_+(q)$. The value of $\sigma(q_+)$ appearing in the inhomogeneous term $f(q)$ can be calculated from $\tilde{\tau}(x)$ by

$$\sigma(q_+) = \tau_+(0+) = -2\pi i \lim_{x\to\infty} x\tilde{\tau}_+(x). \tag{3.19}$$

From the Bethe ansatz equations (2.14) we have the conditions

$$\int_{q_i^{(k)}}^{\infty} \frac{\mathrm{d}q}{2\pi} \sigma_k(q) = I_\infty^{(k)} - I_i^{(k)}, \qquad 2\pi I_\infty^{(k)} = z_k(\infty), \tag{3.20}$$

for $k=1$ and $q_i = q_+$ we have

$$\tilde{\tau}_+(0) = I_\infty - I_+. \tag{3.21}$$

Analogously to eqs. (3.11) and (3.19)–(3.21) we can derive equations for the function $\tau_-(q) = \sigma(q_- - q)$.

We are now in a position to calculate the eigenvalue of the transfer matrix by means of eqs. (2.24), (3.8), (3.16) and (3.19)–(3.21) up to corrections $O(1/N)$ which vanish faster than $1/N$ for $N \to \infty$. In appendix C the general result for arbitrary excitations is derived. Appendix D contains a proof that all the δ's can be neglected and that the contribution from the Δ-terms cancel in the final result, although they are of the same order as those we are looking for.

Here we consider the simple case that there are no complex roots ($\rho_k = 0$ for $k > 1$) and no real roots outside the interval $[q_-, q_+]$. This means $f_1 = 0$ in eq. (3.13). In the following all relations with $\cong$ are true only up to Δ-terms which are not written down in this section*.

In the present case only f_0 contributes to f. Therefore, the integral in eq. (3.16) can be calculated

$$\tilde{\tau}_+(x) \cong G^+(x)\left(A\frac{\frac{1}{2}i\nu}{x+\frac{1}{2}i\nu} - \tfrac{1}{2} - B(ix+g)\right) + \tfrac{1}{2} + Bix, \tag{3.22}$$

where

$$A = \frac{N}{\pi} \mathrm{e}^{-\nu q_+/2} G^+\left(\tfrac{1}{2}i\nu\right), \qquad B = \tfrac{1}{6}\pi\frac{1}{\sigma(q_+)}, \tag{3.23}$$

and g is defined by the asymptotic expansion of $G^+(x)$ for large x which follow from the definition eq. (3.18)

$$G^+(x) = 1 + igx^{-1} - \tfrac{1}{2}g^2x^{-2} + O(x^{-3}). \tag{3.24}$$

* Analogous formulas in ref. [6] are not completely correct.

From eqs. (3.19), (3.22) and (3.24) we find

$$\tfrac{1}{12} \cong \left(\tfrac{1}{2} + gB\right)^2 - \nu AB . \tag{3.25}$$

The Bethe ansatz equation in the form of eq. (3.21) implies

$$A \cong \tfrac{1}{2} + gB + \frac{1}{G^+(0)}\left(I_\infty - I_+ - \tfrac{1}{2}\right). \tag{3.26}$$

We now can calculate the finite-size correction to the eigenvalue of the transfer matrix produced by the upper bound of the "Dirac sea" of real roots. After a simple algebra using eqs. (3.22), (3.23), (3.25) and (3.26) we obtain

$$e^{\nu q_+/2}\left(1 - B\nu - 2\tilde{\tau}_+\left(\tfrac{1}{2}i\nu\right)\right) \cong \frac{\pi}{N}\left\{\tfrac{1}{12} - \tfrac{1}{2}\frac{\nu}{\nu-1}\left(I_\infty - I_+ - \tfrac{1}{2}\right)^2\right\} + o(N^{-1}), \tag{3.27}$$

where we have used

$$(1/G^+(0))^2 = 1 + \tilde{J}(0) = \tfrac{1}{2}\frac{\nu}{\nu-1}, \tag{3.28}$$

which follow from eq. (3.18). Together with an analogous expression due to the lower bound q_- of the real roots, we find from eqs. (2.24), (3.8) and (3.27) for the eigenvalue up to corrections which vanish faster than $1/N$ for large N that

$$\begin{aligned}\log \lambda_A(\theta, \gamma) = &-Nf_\infty + \tfrac{1}{2}i\pi\left((I_\infty - I_+) - (I_- - I_{-\infty})\right) - i\kappa\gamma \\ &+ \frac{\pi}{N}\left\{\tfrac{1}{6}\sin\nu\theta - \tfrac{1}{2}i\frac{\nu}{\nu-1}\right. \\ &\left.\times\left(e^{-i\nu\theta}\left(I_\infty - I_+ - \tfrac{1}{2}\right)^2 - e^{i\nu\theta}\left(I_{-\infty} - I_- - \tfrac{1}{2}\right)^2\right)\right\}.\end{aligned} \tag{3.29}$$

The numbers $I_{\pm\infty} = (1/2\pi)z_1(\pm\infty)$ are given by eqs. (2.15)

$$I_\infty = N - (\Sigma - \kappa)/\nu, \qquad I_{-\infty} = n + (\Sigma + \kappa)/\nu. \tag{3.30}$$

The numbers $I_\pm$ depend on the state under consideration

$$\begin{aligned} I_+ &= N - \tfrac{1}{2} - h_+, \qquad h_+ = 0, 1, \dots, \\ I_- &= n + \tfrac{1}{2} + h_-, \qquad h_- = 0, 1, \dots, \end{aligned} \tag{3.31}$$

where the $h_\pm$ are the number of holes "near to $\pm\infty$". For the ground state which corresponds to the maximal eigenvalue of τ, i.e. to the maximal number of real roots we have $h = h_+ + h_- = 0$. The "spin" of the Bethe state

$$\Sigma = \tfrac{1}{2}N - n = \tfrac{1}{2}(h_+ + h_-) = 0, 1, \dots \tag{3.32}$$

assumes integer values (N even). The constant κ depends on the model (for the ordinary six-vertex model $\kappa = 0$ holds). Note that there might be some restriction on the number $h_\pm$ if Σ and κ are large and ν is small. These restrictions come from the condition that the "last" roots should not be shifted to $\pm\infty$. They will not relevant in this paper since we are interested in low-energy excitations with $\Sigma = 0$ and $\kappa = 0, 1$.

Inserting eqs. (3.30) and (3.31) in eq. (3.29) we obtain a formula for the eigenvalue of the transfer matrix in terms of the holes $h_\pm$. In appendix C the general formula for arbitrary strings is derived

$$\begin{aligned}\log \lambda_A(\theta, \gamma) = {} & -Nf_\infty + \tfrac{1}{2}i\pi\Big(h_+ - h_- + 2\sum_k n_k(k-1)\Big) \\ & + \frac{\pi}{N}\Big\{\tfrac{1}{6}\sin\nu\theta - \tfrac{1}{2}i\frac{1}{\nu(\nu-1)}\Big(\mathrm{e}^{-i\nu\theta}\big(\nu\hat{h}_+ - \Sigma + \kappa\big)^2 \\ & \qquad - \mathrm{e}^{i\nu\theta}\big(\nu\hat{h}_- - \Sigma - \kappa\big)^2\Big) \\ & \qquad - 2i\big(K\,\mathrm{e}^{-i\nu\theta} - \bar{K}\,\mathrm{e}^{i\nu\theta}\big)\Big\} + \mathrm{O}(1/N),\end{aligned} \tag{3.33}$$

where

$$\hat{h}_\pm = h_\pm - 2\sum_k n_\mp^{(k)}(k-1), \tag{3.34}$$

and $n_\pm^{(k)}$ is the number of k-strings "near to $q_\pm$"*. The integers K and $\bar{K}$ depend on the complex roots and the real roots outside $[q_-, q_+]$. For the case considered above we have $\hat{h}_\pm = h_\pm$ and $K = \bar{K} = 0$.

4. Conformal properties of six-vertex and Potts models

The q-state Potts models are a generalizations of the Ising model where the spin at each site can accept q values instead of two. For $q = 2, 3, 4$ the models possess a critical point which we consider in the following. The partition function on a cylinder [14] can be written in terms of a six-vertex model with a "seam" from the

* The k-strings which do not approach $\pm\infty$ like the last real roots $q_\pm$ as $N \to \infty$ will not affect the $1/N$ corrections.

bottom to the top of the cylinder. This seam carries extra Boltzmann weights which produce an extra phase factor in the Bethe ansatz equations (2.4) and the eigenvalue equation (2.5). These modifications are obtained for $\kappa = 1$. The number q ($q = 2, 3, 4$) is related to the six-vertex coupling $\gamma = \pi/\nu$ ($\nu = 4, 6, \infty$) by

$$q = 4\cos^2\gamma . \tag{4.1}$$

In the following the partition function Z_κ and other formulas apply for $\kappa = 0$ to the ordinary six-vertex model and for $\kappa = 1$ to the Potts model

$$Z_0 = Z^{6-\mathrm{vertex}}, \qquad Z_1 = Z^{\mathrm{Potts}} . \tag{4.2}$$

In sect. 3 the eigenvalue of the transfer matrix has been calculated including finite-size corrections for a large N. Using these results for the ground state, where no holes and complex roots are present, we find that

$$\log \lambda_{A,\max} \approx -N f_\infty + \frac{1}{N}\frac{\pi}{6}\left(1 - \frac{6\kappa^2}{\nu(\nu-1)}\right)\sin \nu\theta . \tag{4.3}$$

The vertex weights (2.2) fulfil $a = b$ for $\nu\theta = \frac{1}{2}\pi$ where we expect isotropy and conformal invariance in the scaling limit. Comparing eq. (4.3) and Cardy's formula (1.8) we conclude that the central charges of the Virasoro algebra of the six-vertex and the q-state Potts models are

$$c^{6-\mathrm{vertex}} = 1 \tag{4.4a}$$

$$c^{\mathrm{Potts}} = 1 - 6/\nu(\nu-1) . \tag{4.4b}$$

We have obtained these results previously in ref. [1] by method (a) mentioned in sect. 1. These results have been expected from renormalization group arguments (see, e.g., ref. [17]).

Excitations possess "holes" and complex roots. We obtain for these states by means of the general formula (3.33) taking Cardy's formulas (1.10) into account*

$$\log \lambda_A - \log \lambda_{A,\max} \approx -\frac{2\pi}{N}\left\{(\Delta + \bar{\Delta})\sin \nu\theta + i(\Delta - \bar{\Delta})\cos \nu\theta\right\} . \tag{4.5}$$

The primary fields correspond to low-energy excitations with a small number of holes $h_\pm$ "near to the last roots" $q_\pm$ which approach $\pm\infty$ for $N \to \infty$. In addition, there may be a number $n_\pm^{(k)}$ of k-strings "near to $q_\pm$". For these configurations of

* Note that the imaginary term correspond to the momentum since $\tau(\theta = 0) = \exp(-iP)$.

roots we obtain from eq. (3.33) for the six-vertex model*

$$\Delta = \frac{\left(\nu\hat{h}_{+} - \Sigma\right)^2}{4\nu(\nu-1)} + K, \qquad \bar{\Delta} = \frac{\left(\nu\hat{h}_{-} - \Sigma\right)^2}{4\nu(\nu-1)} + \bar{K}, \tag{4.6a}$$

and for the Potts model

$$\Delta = \frac{\left(\nu\hat{h}_{+} - \Sigma + \kappa\right)^2 - 1}{4\nu(\nu-1)} + K, \qquad \bar{\Delta} = \frac{\left(\nu\hat{h}_{-} - \Sigma - \kappa\right)^2 - 1}{4\nu(\nu-1)} + \bar{K}, \tag{4.6b}$$

where $\hat{h}_{\pm}$ is given by eq. (3.34), $\Sigma = \frac{1}{2}h - \Sigma_k n_k(k-1)$ is the "spin"** of the Bethe state and $h(n_k)$ is the total number of holes (k-strings). In appendix C it is shown that we get minimal values for Δ and $\bar{\Delta}$ (i.e. the integers K and $\bar{K}$ on the r.h.s. of eqs. (4.6) vanish) if some conditions are fulfilled. There are no holes inside $[q_-, q_+]$, $n_+^{(k)} = 0$ for all k, $n_-^{(k)} = 1$ for only one k and the number of holes have to fulfil $h_+ = h_- = k - 1$. For the six-vertex model we obtain the conformal dimensions from eqs. (4.6a). These results agree with Kadanoff's work on the Gaussian model (see, e.g. ref. [18]).

The extra phase factors proportional to κ imply (see appendix C) that for the minimal Δ and $\bar{\Delta}$ only the k-strings with

$$k = 2, 3, \ldots, \nu - \kappa - 1 \tag{4.7}$$

contribute. Hence, for the q-state Potts model ($\kappa = 1$) the $\nu - 1$-string does not contribute. A similar argument holds for the "antipseudoparticle". We consider fixed boundary conditions at the top and the bottom of the cylinder. Therefore, we have spin $\Sigma = 0$. Comparing eqs. (4.6b) for $\Sigma = 0$ with the Kac formula (1.5) we find for the Potts model the conformal dimensions

$$\Delta = \bar{\Delta} = h_{k,1} \qquad k = 1, 2, \ldots, \nu - 2. \tag{4.8}$$

These correspond to the "energy operators" of the Potts model (cf. refs. [12, 19, 20, 26]).

The equivalence of the Potts model and a six-vertex model with "seam" for $\kappa = 1$ holds for the partition functions (cf. eq. (4.2)) but not for the operator content. The order parameter of the Potts model, for example, appears in a six-vertex model with $\kappa = \frac{1}{2}\nu$ (cf. [12]). If we consider a partition function of a generalized model

$$Z = Z_1 + \sum Z_\kappa, \tag{4.9}$$

* These formulas agree with results obtained for the Heisenberg model [5].

** Σ must not be confused with the "spin" s of the field operator (cf. eq. (1.10)).

where the sum extends over some values of $\kappa \in [2, \nu - 1]$, we obtain the dimensions corresponding to Bethe states with a k-string

$$\Delta = \bar{\Delta} = h_{k+\kappa-1,\kappa}, \qquad k = 1, \dots, \nu - \kappa - 1, \qquad 1 \leqslant \kappa \leqslant \nu - 1. \tag{4.10}$$

Note that the ground state always belongs to Z_1. The "magnetization" operators of the $q = 3$-state Potts model ($\nu = 6$) correspond to $\kappa = \frac{1}{2}\nu = 3$ (cf. ref. [20]). Their dimensions are therefore (in agreement with numerical estimates [20] and conjectures [12, 19])

$$\Delta = \bar{\Delta} = h_{k+2,3}, \qquad k = 1, 2. \tag{4.11}$$

The $1/q$-parafermion operators [21] correspond to "boundary conditions" $\kappa = \nu/q = 2$ for $q = 2, 3$ [21, 20]. The Bethe states have spin $\Sigma = 1$ and a symmetric hole configuration $h_+ = h_- = 1$. For $\Sigma \neq 0$ the integers on the r.h.s. of eqs. (4.6) only vanish, if there are no higher strings. Hence we obtain from eqs. (4.6b) the scaling dimensions and spin

$$d = \Delta + \bar{\Delta} = h_{21} + h_{23} = \frac{\nu - 1}{2\nu} + \frac{\nu^2 - q^2}{2q^2\nu(\nu - 1)},$$

$$s = \Delta - \bar{\Delta} = h_{21} - h_{23} = 1/q, \tag{4.12}$$

in agreement with refs. [20, 21, 26]. Finally, let us look for the "chiral" operators of the 3-state Potts model (cf. refs. [12, 19]). Their conformal dimensions are given by the "boundary conditions" $\kappa = \frac{1}{2}\nu = 3$ and Bethe states with $\Sigma = 2$ or four holes. For $h_+ = 0, 1, 2, 3$ and $h_- = 4 - h_+$ one finds from eqs. (4.6b), respectively, that

$$\Delta = h_{11}, h_{21}, h_{31}, h_{41} = 0, \tfrac{2}{5}, \tfrac{7}{5}, 3,$$

$$\bar{\Delta} = h_{15}, h_{25}, h_{35}, h_{45} = 3, \tfrac{7}{5}, \tfrac{2}{5}, 0.$$

At the end, let us consider a partition function given by eq. (4.9) where the sum extends over all $\kappa = 2, \dots, \nu - 1$. The dimension formula (4.10) obviously describes the dimensional spectrum of the RSOS models [22], in the language of ref. [23] the $(A_\nu, A_{\nu-1})$ series of conformal quantum field theories. For this interpretation of eq. (4.9) the number κ is not connected to extra weights on a "seam" or to "boundary conditions", but it is related [25] to a "Θ-vacuum" like angle appearing in the construction of Bethe states [13]. The RSOS models of ref. [22] can be generalized [24] to models corresponding to all modular invariant conformal quantum field theories with $c < 1$ [23]. All these models are related to modified six-vertex models. Results for these models will be published elsewhere [25].

Appendix A

In this appendix we derive an Euler–Maclaurin like formula for the difference of an integral and a sum as appearing in eqs. (3.1) and (3.10)

$$\int_{I_-}^{I_+} \mathrm{d}I\, F(I)\Big(1-\sum_i \delta(I-I_i)\Big). \tag{A.1}$$

where $\{I_i\}=(\boldsymbol{Z}+\frac{1}{2})\cap[I_-, I_+]$ and the integration bounds $I_\pm \in \boldsymbol{Z}+\frac{1}{2}$ are understood as $I_\pm \pm 0$. By means of partial integrations we obtain

$$\int_{I_-}^{I_+} \mathrm{d}I\, F(I)\, g_0(I) = \sum_{r=0}^{s}\left[(-1)^r F^{(r)}(I)\, g_{r+1}(I)\right]_{I_-}^{I_+}$$

$$+(-1)^s \int_{I_-}^{I_+} \mathrm{d}I\, F^{(s+1)}(I)\, g_{s+1}(I), \tag{A.2}$$

where $g_0(I)=1-\sum_i \delta(I-I_i)$ and $g'_{r+1}=g_r$ are periodic functions with period 1. A convenient normalization for the functions g_r is

$$\int_{-1/2}^{1/2} \mathrm{d}I\, g_r(I)=0. \tag{A.3}$$

The first three functions for $|I|<\frac{1}{2}$ and their values at $\pm\frac{1}{2}$, ± 0 are given by

$$\begin{aligned} g_1(I)&=I, & g_1(\pm\tfrac{1}{2})&=\mp\tfrac{1}{2},\\ g_2(I)&=\tfrac{1}{2}I^2-\tfrac{1}{24}, & g_2(\pm\tfrac{1}{2})&=\tfrac{1}{12},\\ g_3(I)&=\tfrac{1}{6}I^3-\tfrac{1}{24}I, & g_3(\pm\tfrac{1}{2})&=0. \end{aligned} \tag{A.4}$$

For $s=2$ we obtain the formula used in sect. 3

$$\int_{I_-}^{I_+} \mathrm{d}I\, F(I)\Big(1-\sum_i \delta(I-I_i)\Big)$$

$$=-\tfrac{1}{2}\big(F(I_+)+F(I_-)\big)-\tfrac{1}{12}\big(F'(I_+)-F'(I_-)\big)-\int_{I_-}^{I_+} \mathrm{d}I\, F'''(I)\, g_3(I). \tag{A.5}$$

Appendix B

In this appendix we prove that the solution of the Wiener–Hopf integral equation (3.11)

$$\tau_+(q)=f(q)-\int_0^\infty \frac{\mathrm{d}q}{2\pi} J(q-q')\,\tau_+(q'), \tag{B.1}$$

for the function $f(q)$ defined by eqs. (3.12) is given by eq. (3.16)

$$\tilde{\tau}_+(y) = G^+(y)\frac{1}{2\pi i}\int \frac{\mathrm{d}z}{z-y-i\varepsilon} G^-(z)\tilde{f}(z). \tag{B.2}$$

Writing f and J as Fourier transformations we transform eq. (B.1) to

$$\begin{aligned}\tilde{\tau}_+(x) &= \int_0^\infty \frac{\mathrm{d}q}{2\pi} e^{iqx}\tau_+(q) \\ &= \frac{1}{2\pi i}\int \frac{\mathrm{d}y}{y-x-i\varepsilon}\left(\tilde{f}(y) - \tilde{J}(y)\tilde{\tau}_+(y)\right).\end{aligned} \tag{B.3}$$

Using eq. (B.2) the second term on the r.h.s. of eq. (B.3) becomes after interchanging of the integrations

$$\begin{aligned}&\frac{1}{2\pi i}\int \mathrm{d}z\, G^-(z)\tilde{f}(z)\frac{1}{2\pi i}\int \frac{\mathrm{d}z}{(y-x-i\varepsilon)(z-y-i\varepsilon)} G^+(y)\tilde{J}(y) \\ &= \frac{1}{2\pi i}\int \frac{\mathrm{d}z}{z-x-i\varepsilon} G^-(z)\tilde{f}(z)\left(1/G^-(z) - G^+(x)\right),\end{aligned} \tag{B.4}$$

where $G^+(y)\tilde{J}(y) = 1/G^-(y) - G^+(y)$ has been used and the fact that $G^{\pm}(\pm y) \neq 0$ and is analytic in the upper half plane, and $G^{\pm}(\infty) = 1$. Inserting eq. (B.4) in eq. (B.3) one finds that eq. (B.2) solves the Wiener–Hopf equation (B.1).

Appendix C

The eigenvalues of the transfer matrix including $1/N$ corrections are determined for arbitrary low-energy Bethe states.

In this appendix we neglect the δ-terms which are of higher order. Further, we do not write the Δ-terms which cancel in the final result. A justification for both of these procedures is given in appendix D. We calculate the eigenvalue given by eq. (2.24)

$$\log \lambda_A(\theta, \gamma) = -Nf_\infty + \int \frac{\mathrm{d}q}{2\pi}\varepsilon(q)\varphi(q) + i\pi\sum_k n_k(k-1) - i\kappa\gamma, \tag{C.1}$$

by means of the Bethe ansatz equations in the form of eq. (3.20)

$$\int_{q_i^{(k)}}^\infty \frac{\mathrm{d}q}{2\pi}\sigma_k(q) = I_\infty^{(k)} - I_i^{(k)}, \qquad 2\pi I_\infty^{(k)} = z_k(\infty), \tag{C.2}$$

where $\sigma_k(q)$ fulfills eq. (2.21)

$$\sigma_k = \rho_0\delta_{k1} - J_k * \varphi + \sum_{l>1} J_{kl} * \rho_l . \tag{C.3}$$

We consider an interval $[q_-, q_+]$ which does not contain any holes and which grows such that $q_\pm \to \pm\infty$ for $N \to \infty$. The Euler–Maclaurin formula applied to eqs. (C.1) and (C.3) gives, respectively

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}q}{2\pi} \varepsilon(q)\varphi(q) \cong \tfrac{1}{2} i\pi\tilde{\tau}_+(0) - \sum_i{}' \left(\tfrac{1}{2}i\pi - i2\,\mathrm{e}^{-i\nu\theta}\mathrm{e}^{-\nu q_+/2}\right)$$

$$+ i\,\mathrm{e}^{-i\nu\theta}\mathrm{e}^{-\nu q_+/2}\left(1 - B\nu - 2\tilde{\tau}_+\left(\tfrac{1}{2}i\nu\right)\right) + \text{“}q_-\text{-terms”}, \tag{C.4}$$

$$\sigma_k(q) \cong \delta_{k1} N\nu\,\mathrm{e}^{-\nu q/2} + \tfrac{1}{2}J_k(q - q_+) - BJ_k'(q - q_+)$$

$$- \int_0^\infty \frac{\mathrm{d}q}{2\pi} J_k(q - q_+ - q')\tau_+(q') + \sum_{l,j}{}' J_{kl}\left(q - q_j^{(l)}\right). \tag{C.5}$$

The symbol Σ' means that for $k = 1$ the sum extends only over those real roots q_i which are not in $[q_-, q_+]$. Relations with $\cong$ are true only up to Δ-terms. For $k = 1$, eq. (C.5) gives a Wiener–Hopf integral equation for $\tau_+(q) = \sigma(q + q_+)$

$$\tau_+(q) = f(q) - \int_0^\infty \frac{\mathrm{d}q'}{2\pi} J(q - q')\tau_+(q'), \tag{C.6}$$

where f is given by eq. (3.13)

$$f(q) \cong f_0(q) + f_1(q),$$

$$f_0(q) = N\nu\,\mathrm{e}^{-\nu q_+/2}\mathrm{e}^{-\nu q/2} + \tfrac{1}{2}J(q) - BJ'(q),$$

$$f_1(q) = \sum_{k,i}{}' J_k\left(q + q_+ - q_i^{(k)}\right). \tag{C.7}$$

An analogous integral equation as for τ_+ holds for τ_-. The solution of the Wiener–Hopf equation (C.2) is given by eqs. (3.16) or (B.2). The contribution from f_0 to τ_+ has been calculated in eq. (3.22)

$$\tilde{\tau}_{0+}(x) \cong G^+(x)\left(A\frac{\tfrac{1}{2}i\nu}{x + \tfrac{1}{2}i\nu} - \tfrac{1}{2} - B(ix + g)\right) + \tfrac{1}{2} + Bix, \tag{C.8}$$

where

$$A = \frac{N}{\pi} e^{-\nu q_+/2} G^+\left(\tfrac{1}{2} i\nu\right), \qquad B = \tfrac{1}{6}\pi \frac{1}{\sigma(q_+)} . \tag{C.9}$$

The contribution from f_1 is

$$\tilde{\tau}_{1+}(x) = \sum_{k,i}{}' \alpha_i^k(x)$$

$$\alpha_i^k(x) = G^+(x) \frac{1}{2\pi i} \int \frac{\mathrm{d}y}{y - x - i\varepsilon} G^-(y) \mathrm{e}^{iy(q_i^{(k)} - q_+)} J_k(y) . \tag{C.10}$$

From eqs. (3.19) and (3.21)

$$\sigma(q_+) = -2\pi i \lim_{x\to\infty} x\tilde{\tau}_+(x), \qquad \tilde{\tau}_+(0) = I_\infty - I_+ , \tag{C.11}$$

we derive the formulas analogous to eqs. (3.26) and (3.25)

$$\tfrac{1}{12} \cong \left(\tfrac{1}{2} + gB\right)^2 - \nu AB + 2e, \qquad A \cong \tfrac{1}{2} + gB + d - c, \tag{C.12}$$

where the following abbreviations have been introduced

$$d = \tilde{\tau}_+(0)/G^+(0) = \left(I_\infty - I_+ - \tfrac{1}{2}\right)/G^+(0),$$

$$c = \sum_{k,i}{}' c_i^k, \qquad c_i^k = \alpha_i^k(0),$$

$$e = \sum_{k,i}{}' e_i^k, \qquad e_i^k = iB \lim_{x\to\infty} x\alpha_i^k(0) . \tag{C.13}$$

Hence, we find, as a generalization of eq. (3.27), that

$$\mathrm{e}^{\nu q_+/2}\left(1 - B\nu - 2\tilde{\tau}_+\left(\tfrac{1}{2} i\nu\right)\right) \cong \frac{\pi}{N}\left\{\tfrac{1}{12} - (c-d)^2 - 2e - 2f\right\}, \tag{C.14}$$

where

$$f = \sum_{k,i}{}' f_i^k , \qquad f_i^k = \frac{A}{G^+\left(\tfrac{1}{2} i\nu\right)} \alpha_i^k\left(\tfrac{1}{2} i\nu\right) . \tag{C.15}$$

In order to apply the Bethe ansatz equations in the form of eq. (C.2) we use eq. (C.3) and express the function $\sigma_k(q)$ for large q in terms of $\tau_+(q)$. By means of

Fourier transformation the three terms in eq. (C.5) involving J_k we write

$$\int_{-\infty}^{\infty} \mathrm{d}x \tilde{J}_k(x) \mathrm{e}^{-ix(q-q_+)}\left(\tfrac{1}{2} + Bix - \tilde{\tau}_+(x)\right). \tag{C.16}$$

Inserting the solution for τ_+ given by eqs. (C.8) and (C.10) we obtain from eqs. (C.2), (C.5) and (C.16) after some computations

$$\begin{aligned} I_\infty^{(k)} - I_i^{(k)} &= \int_{q_i^{(k)}}^{\infty} \frac{\mathrm{d}q}{2\pi} \sigma_k(q) \\ &= \delta_{k1} \frac{N}{\pi} \nu \, \mathrm{e}^{-\nu q_i/2} - \tilde{J}_k(0) G^+(0) d - e_i^k - f_i^k - \sum_{l,j}{}' \left(g_{ij}^{kl} + a_{ij}^{kl}\right), \end{aligned} \tag{C.17}$$

where

$$g_{ij}^{kl} = \frac{1}{2\pi i} \int \frac{\mathrm{d}x}{x - i\varepsilon} \mathrm{e}^{ix(q_+ - q_i^{(k)})} \alpha_j^l(x), \qquad a_{ij}^{kl} = \int_{q_i^{(k)}}^{\infty} \frac{\mathrm{d}q}{2\pi} J_{kl}\left(q - q_j^{(l)}\right). \tag{C.18}$$

One finds using eq. (C.10) and symmetry with respect to x and y and the symmetry of $J_{kl}(q)$, respectively that

$$g_{ij}^{kl} + g_{ji}^{lk} = -c_i^k c_j^l, \tag{C.19}$$

$$\begin{aligned} a_{ij}^{kl} + a_{ji}^{lk} = \tilde{J}_{kl}(0) &= \delta_{kl} - 2\min(k,l) - 2kl + 2k + 2l - k\delta_{l1} - l\delta_{k1} \\ &\quad + \left(G^+(0)\right)^2 \left(k + \tilde{J}_k(0)\right)\left(l + \tilde{J}_l(0)\right). \end{aligned} \tag{C.20}$$

Summing eq. (C.17) over all complex roots and real roots above q_+ we find after some algebra

$$\tfrac{1}{2}(c-d)^2 + e + f - \frac{N}{\pi} \sum_i{}' \mathrm{e}^{-\nu q_i/2} = \tfrac{1}{2}\hat{d}^2 + K, \tag{C.21}$$

where (with $\hat{h}_+$ defined by eq. (3.34))

$$\begin{aligned} \hat{d} &= d - G^+(0) \sum_k n_+^{(k)} \left(k + \tilde{J}_k(0)\right) \\ &= \frac{1}{\nu}\left(\nu \hat{h}_+ - \Sigma + \kappa\right) / G^+(0). \end{aligned} \tag{C.22}$$

The integer K can be expressed in terms of $n_+^{(k)}$, the number of k-strings "near to

q_+" (for $k=1$ only those above q_+ contribute) and h_+, the number of holes above q_+,

$$K=\sum_k\left\{\tfrac{1}{2}n_+^{(k)}\left(n_+^{(k)}-1\right)-\sum_i{}' l_i^{(k)}\right.$$
$$\left.+n_+^{(k)}k\left(h_+-\sum_l n_+^{(l)}(l-1)\right)+\sum_{l>k}n_+^{(k)}n_+^{(l)}(l-k)\right\}. \qquad (C.23)$$

For the derivation of eqs. (C.21)–(C.23) the following have been used (cf. eqs. (3.30 and 31)

$$I_\infty^{(k)}=N-\frac{1}{\nu}k(\Sigma-\kappa),\qquad I_i^{(k)}=N-\tfrac{1}{2}-l_i^{(k)},\qquad l_i^{(k)}\geqslant 0 \qquad (C.24)$$

$$I_+=N-\tfrac{1}{2}-h_+-n_+^{(1)}. \qquad (C.25)$$

Inserting eqs. (C.14) and (C.21) in eq. (C.4) we obtain

$$\int_{-\infty}^{+\infty}\frac{\mathrm{d}q}{2\pi}\varepsilon(q)\varphi(q)\cong\tfrac{1}{2}i\pi\left(I_\infty-I_+-n_+^{(1)}\right)$$
$$+\frac{\pi}{N}i\,\mathrm{e}^{-i\nu\theta}\left\{\tfrac{1}{12}-\hat{d}^2-2K\right\}+\text{“}q_-\text{-terms”} \qquad (C.26)$$

Together with the corresponding contributions from q_- and eq. (C.1) one derives the final result eq. (3.33). The integer $\bar{K}$ is given by eq. (C.23) where $n_+^{(k)}$, $h_+^{(k)}$, $l_i^{(k)}$ are replaced by $n_-^{(k)}$, $h^{(k)}$, $\bar{l}_i^{(k)}$, respectively.

At the end of this appendix we investigate conditions which select states with $K=\bar{K}=0$, i.e. excitations corresponding to primary fields. One can separate in eq. (C.23) contributions from the real roots and find

$$K^{(1)}=\sum_{j=1}^{h_+}i_j-\tfrac{1}{2}h_+(h_++1), \qquad (C.27)$$

if the holes are located at

$$I=N-\tfrac{1}{2}-i_j,\qquad i_j\geqslant 0. \qquad (C.28)$$

Obviously, $K^{(1)}\geqslant 0$ and $K^{(1)}=0$ if there are only holes outside the bunch of real roots. Discussing $K^{(k)}$ for $k>1$ one must take into account the restriction

$$I_i^{(k)}+\bar{I}_i^{(k)}<I_\infty^{(k)}-I_{-\infty}^{(k)}. \qquad (C.29)$$

After inspection one finds that K and $\bar{K}$ only vanish if there is, at most, one k-string ($k > 1$). If we assume

$$n^{(l)} = \delta_{kl}, \tag{C.30}$$

the following relations in addition must hold for $\kappa > 0$

$$h_+ = h_- = k - 1, \qquad n_+^{(k)} = 0, \qquad n_-^{(k)} = 1. \tag{C.31}$$

Note that for $\kappa > 0$ the extra phase in eq. (2.13) shifts the string near to q_-. For $\kappa = 0$ because of symmetry the string will be at 0 which means $n_\pm^{(k)} = 0$.

If $\kappa > 0$ not all strings are allowed, too large ones are shifted to $-\infty$. For spin $\Sigma = 0$ states the argument is as follows. In the situation characterized by eqs. (C.30) and (C.31) one has

$$I_1^{(k)} = N - \tfrac{1}{2}. \tag{C.32}$$

But there is a lower bound for this number given by

$$\min z_k\big(q_1^{(k)}\big) = z_k(-\infty) + \Phi_{kk}(0) - \Phi_{kk}(-\infty) = 2\pi\big(N - \tfrac{1}{2} - (k/\nu)(\nu - \kappa - k)\big). \tag{C.33}$$

Therefore we have the condition

$$k < \nu - \kappa. \tag{C.34}$$

This means that for $\kappa = 1$ the $\nu - 1$-strings are excluded. An analogous argument excludes the "antipseudoparticles".

Appendix D

In this appendix we prove that the δ-terms given by eqs. (3.9) and (3.14) can be neglected in eqs. (3.8) and (3.12). In addition, we show that the contributions from the Δ-terms defined by eqs. (3.4) and (3.15) cancel in the final formulas (3.29) and (3.33) for the finite size corrections of the eigenvalue of the transfer matrix.

In order to apply the Euler–Maclaurin formula we have chosen the bounds $I_\pm = I(q_\pm)$ arbitrarily, only fulfilling the conditions

$$\text{holes} \notin [q_-, q_+], \qquad q_\pm \to \pm\infty \qquad \text{for } N \to \infty.$$

The maximal possible $I_\pm$ are of order $O(N)$ for large N. But since the formula for the eigenvalue (3.33) does not depend on the bounds $I_\pm$ used in the derivation, we may also take smaller ones. In the following it is shown that all δ-terms can be neglected if the $I_{\pm}$ are large enough.

For large N and $q \ll q_{+,\max}$ we have $\sigma(q) \approx \rho_0(q)$. This implies that for $2/\nu \ll q \ll q_+$ or $\frac{1}{2}N \ll I \ll N$ the following asymptotic behaviour holds

$$\left.\begin{aligned} \sigma(q) &= \mathrm{O}(N - I(q)) \\ \frac{\mathrm{d}q}{\mathrm{d}I} &= \frac{2}{\nu}\frac{1}{N-I} + \mathrm{O}\big((N-I)^{-2}\big) \\ N\mathrm{e}^{-\nu q(I)/2} &= \mathrm{O}(N-1) \end{aligned}\right\} \quad \text{for} \quad N - I \to \infty . \tag{D.1}$$

Hence, we can estimate the δ-terms for large N and large $N-I$

$$\delta_1 = N^{-3}\mathrm{O}\big((N-I)^4\big), \qquad \delta_2 = N^{-3}\mathrm{O}\big((N-I)^3\big), \qquad \delta_3 = N^{-3}\mathrm{O}\big((N-I)^2\big), \tag{D.2}$$

and for $q > 0$

$$\delta_4, \delta_5 = N^{-2/(\nu-1)}\mathrm{O}\big((N-I)^{2/(\nu-1)}\big), \qquad \delta_6 = N^{-2/(\nu-1)}\mathrm{O}\big((N-I)^{1+2/(\nu-1)}\big). \tag{D.3}$$

If we take $N - I_+ = \mathrm{O}(N^{-\alpha})$ with $0 < \alpha < 1/\nu$ we find that $N\delta_{1,2,3}$ and $(N - I_+)\delta_{4,5,6}$ vanish for large N. Therefore, we can neglect $\delta_{1,2,3}$ in eq. (3.8) compared with the terms $\mathrm{O}(N^{-1})$ we are interested in. The largest contribution from $\delta_{4,5,6}$ appears in eq. (C.21). It is of the order $(N - I_+)\delta_{4,5,6}$ which also can be neglected. Altogether, we have found that the δ-terms do not contribute if I_+ is large enough, e.g., if

$$N - I_+ = \mathrm{O}(N^{-\alpha}), \qquad 0 < \alpha < 1/\nu . \tag{D.4}$$

Now follows the proof that the Δ-terms cancel in the final formula. We have observed that the formula for the eigenvalue (3.33) (where K is given by eqs. (C.23) and (C.27)) does not depend on the bounds $I_{\pm}$, but it depends only on the "physical" quantities as locations of holes and strings and their numbers. This is obviously a trivial result, since the bounds where choosen arbitrarily. Therefore, if the $I_{\pm}$ are large enough such that the δ-terms can be neglected, the sum of the Δ-terms is a constant as a function of $I_{\pm}$ ★. In the following we show that this constant vanishes.

★ This argument is due to Woynarovich.

There is no reason that the Δ-terms for $I_+ = O(N)$ do not contribute. In fact numerical calculations suggest that they give finite contributions to eqs. (3.22), (3.25), (3.26) and (3.27). In the following I will show that the Δ-terms vanish if I_+ is small enough, e.g. if eq. (D.4) holds for $\alpha > 0$. This means the sum of the Δ-terms in the final formulas, eqs. (3.29) and (3.33) is zero independently of I_+.

It is easy to see that the contributions to the integrals defining $\Delta_{1,2}$ from integration regions corresponding to finite q are of order $O(N^{-3})$. Therefore, we have to investigate for $q_0 \gg 2/\nu$

$$\hat{\Delta}_{1,2} = \int_{I(q_0)}^{I_+} \mathrm{d}I \, (\mathrm{d}/\mathrm{d}I)^3 F_{1,2}(q(I)) g_3(I), \tag{D.5}$$

where

$$F_1(q(I)) = \varepsilon(q(I)),$$
$$F_2(q(I)) = J(q + q_+ - q(I)), \qquad q > 0. \tag{D.6}$$

Since g_3 is bounded we find

$$\hat{\Delta}_{1,2} = O\big((\mathrm{d}/\mathrm{d}I)^2 F_{1,2}(q(I_+))\big) = O\big((N - I_+)^{-2} F_{1,2}(q(I_+))\big), \tag{D.7}$$

where eq. (D.1) has been used. With $F_1(q(I)) \propto \exp(-\frac{1}{2}\nu q(I)) = N^{-1} O(N - I)$ and $F_2(q(I)) = O(1)$ we find that

$$\Delta_1 = N^{-1} O\big((N - I_+)^{-1}\big), \qquad \Delta_2 = O\big((N - I_+)^{-2}\big). \tag{D.8}$$

Similar arguments as above imply that the Δ terms in the final formulas (3.29) and (3.33) are arbitrarily small if I_+ is taken small enough, e.g. eq. (D.4) with $\alpha > 0$. But since the final formula does not depend on the choice of $I_\pm$ the Δ-terms must cancel.

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