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CONFORMAL INVARIANCE AND INTEGRABLE THEORIES

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Integrable theories on the lattice having zero energy gap exhibit conformal invariance for long distances. It is shown here how to extract its conformal properties (central charge c and scaling dimensions) from the Bethe ansatz equations. The methods here exposed are applied to the six-vertex model and to the critical Potts model.

1. Introduction

Conformal invariance is a powerful concept in several regions of mathematical physics. Statistical mechanical systems at second-order phase transition points possess scale and conformal invariance [1]. For conformal quantum field theories Polyakov [2] proposed a "bootstrap" approach for constructing a complete set of local fields. In two dimensions conformal invariance is a rather strong restriction since the transformation group is infinite dimensional. It is related to the Virasoro algebra [3]

$$[L_n, L_m] = (n-m)L_{n+m} + \tfrac{1}{12}cn(n^2-1)\delta_{n,-m}$$
$$(c = \text{"central charge"}) \tag{1.1}$$

for which a well developed theory exists [4]. This formalism is also useful for string theories [3]. Belavin, Polyakov and Zamolodchikov [5] have solved the conformal "bootstrap" problem in two dimensions for many cases. Under a conformal transformation (in complex coordinates)

$$z = x_1 + ix_2 \to w(z),$$
$$\bar{z} = x_1 - ix_1 \to \bar{w}(\bar{z}) \tag{1.2}$$

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(where w and $\bar{w}$ are analytic functions) primary fields transform like

$$A(z,\bar{z}) \to \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{\Delta}\left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right)^{\bar{\Delta}} A(w,\bar{w}), \tag{1.3}$$

where $d_A = \Delta + \bar{\Delta}$ and $S_A = \Delta - \bar{\Delta}$ are the "scaling dimension" and "spin" of the field A, respectively. In ref. [5] it was shown that for a finite set of primary fields the solution of the conformal "bootstrap" corresponds to an irreducible representation of the Virasoro algebra with central charge

$$c = 1 - \frac{6}{m(m+1)}, \tag{1.4}$$

where m is a rational number. Moreover, Friedan, Qiu and Shenker [6] have shown that unitarity in quantum field theory (equivalent to reflection positivity [7] in statistical systems) restrict the number m to

$$m = \text{integer} \geqslant 2 \qquad \text{or} \qquad c \geqslant 1. \tag{1.5}$$

In addition the authors of [6] compared the critical exponents for some new statistical models with the scaling dimension $\Delta + \bar{\Delta}$ appearing in representations of the Virasoro algebra for $m = 3,4,5,6$. They proposed in this way for the critical q-state Potts model

$$c^{\text{Potts}} = 1 - \frac{6}{\nu(\nu-1)} \begin{cases} \nu = 4 & \text{for } q = 2 \quad \text{(Ising)}. \\ \nu = 6 & \text{for } q = 3 \end{cases} \tag{1.6}$$

With analogous arguments Huse [8] suggested for the "restricted solid on solid" models [9]

$$c^{\text{RSOS}} = 1 - \frac{6}{\nu(\nu-1)}, \tag{1.7}$$

where the "heights" l of the model are restricted to $0 < l < \nu$. By means of renormalization group arguments Blöte, Cardy and Nightingale [10] argued eq. (1.6) for the q-state Potts models with

$$q = 4\cos^2(\pi/\nu) \qquad \text{for } q = 1,2,3,4;\ \nu = 3,4,6,\infty \tag{1.8}$$

and for the six-vertex model

$$c^{6\text{-vertex}} = 1. \tag{1.9}$$

For other investigations of unitary conformal invariant models see also ref. [11].

Since a direct calculation of the central charge c and conformal dimensions of the field for specific models is usually not simple, it seems worthwhile to determine them for integrable models from the Bethe ansatz solution. In this paper a general method to compute the conformal properties of an integrable theory is presented. As concrete applications we obtain explicit results for the six-vertex and q-state Potts models.

For a statistical system on an $M \times N$ lattice in the x-y plane one can introduce a transfer matrix τ (defined on an M-chain) running in y-direction or another one $\hat{\tau}$ (defined on an N-chain) running in x-direction. The partition function can be written as

$$Z_{MN} = \begin{cases} \operatorname{tr} \tau^N = \sum_i \lambda_i^N \\ \operatorname{tr} \hat{\tau}^M = \sum_i \hat{\lambda}_i^M , \end{cases} \tag{1.10}$$

where the sums extend over all eigenvalues of the transfer matrices

$$\begin{aligned} \tau\psi &= \lambda\psi , \\ \hat{\tau}\hat{\psi} &= \hat{\lambda}\hat{\psi} . \end{aligned} \tag{1.11}$$

For a conformal invariant model with periodic boundary condition Cardy [12] has found for $M, N \gg 1$

$$\lambda_{\max}^N \simeq \exp\left(-NMf + \frac{N}{M}\frac{\pi}{6}c \right), \tag{1.12a}$$

$$\hat{\lambda}_{\max}^M \simeq \exp\left(-NMf + \frac{M}{N}\frac{\pi}{6}c \right), \tag{1.12b}$$

where f is the free energy per site and c is the central charge of the Virasoro algebra. For $M \gg N \gg 1$ it follows

$$Z_{MN} \simeq \begin{cases} \exp(-NMf) \sum_i (\lambda_i/\lambda_{\max})^N & (1.13a) \\ \exp\left(-NMf + \frac{M}{N}\frac{\pi}{6}c \right). & (1.13b) \end{cases}$$

For an integrable theory two methods to calculate Z_{MN} on a strip $M \gg N \gg 1$ for large but finite values of N are available:

(a) one considers the transfer matrix τ on an infinite chain ($M \to \infty$) and calculates the large N corrections (due to "low energy" excitations) to the sum in eq. (1.13a), or

(b) one considers the transfer matrix $\hat{\tau}$ on an N-sites chain and calculates the large N corrections to the maximal eigenvalue $\hat{\lambda}_{\max}$ according to eq. (1.13b).

In addition to the "central charge" c also the conformal dimensions of operators of the theory can be determined if one looks for excited states. As it is argued in ref. [12] one has

$$E_n - E_0 = \frac{2\pi x_n}{N} \qquad \text{for } N \to \infty\,, \tag{1.14}$$

where x_n is the conformal dimension of the operator associated to the excitation. This formula applies to energy eigenvalues, so in our context we should consider eigenvalues of $-\ln \hat{\tau}$ in order to use eq. (1.14).

We briefly review in sect. 2 the Bethe ansatz solution for the six-vertex model. The method (a) (to calculate the central charge c) is applied in sect. 3 to the six-vertex model and in sect. 4 to the q-state Potts model. For the six-vertex model also method (b) will be used. This is realized in sect. 5. Conformal dimensions are also calculated in sect. 5.

2. Six-vertex model – Bethe ansatz

The six-vertex model on an $L = M \times N$ square lattice is defined by the partition function

$$Z_{MN} = \sum_{\text{conf}} \prod_{x \in \text{lattice}} \omega_{i(x)}\,, \tag{2.1}$$

where the sum extends over all "bond configurations". Each bond can accept one of two states characterized by arrows. The six allowed configurations at a vertex are depicted in fig. 1 and the corresponding weights are

$$\begin{aligned} &\omega_1 = \omega_2 = 1\,,\\ &\omega_3 = \omega_4 = t = \sin\theta/\sin(\gamma-\theta)\,,\\ &\omega_5 = \omega_6 = r = \sin\gamma/\sin(\gamma-\theta)\,, \end{aligned} \tag{2.2}$$

depending on the "anisotropy" or "spectral parameter" θ and the "coupling" γ.

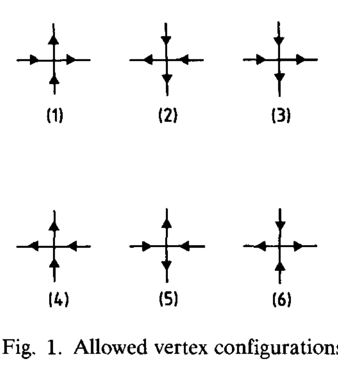

Fig. 1. Allowed vertex configurations.

The partition function with periodic boundary conditions

$$Z_{MN} = \mathrm{tr}\left[\tau(\theta,\gamma)^N\right] \tag{2.3}$$

can be written in terms of a transfer matrix $\tau(\theta,\gamma)$ defined on a periodic chain of length M. By means of the Bethe ansatz method (see e.g. [13]) one obtains the eigenvalues of the transfer matrix

$$\lambda^{(q)} = \lambda_A^{(q)} + \lambda_D^{(q)}, \qquad \lambda_D(\theta,\gamma) = t^M(\theta,\gamma)\lambda_A^*(\gamma-\theta^*,\gamma),$$

$$\lambda_A^{(q)}(\theta,\gamma) = \prod_{i=1}^{m} \exp\left[-i\left(p(q_i+2i\theta)-\pi\right)\right], \tag{2.4}$$

where the numbers $q_1,\ldots,q_m$ $(m=1,\ldots \leqslant \frac{1}{2}M)$ are roots of the "Bethe ansatz equations"

$$\mathrm{e}^{iMp(q_j)} \prod_{\substack{i=1\\ i\neq j}}^{m} \mathrm{e}^{i\phi(q_j-q_i)} = 1\,; \qquad j=1,\ldots,m. \tag{2.5}$$

These equations are quite general. They express the condition that the total phase factor around a period M for "pseudoparticles" of momentum $p(q_j)$ has to be equal to one. This factor consists of the free part $\mathrm{e}^{iMp(q_j)}$ and the scattering parts with respect to all the others $\mathrm{e}^{i\phi(q_j-q_i)}$. For the six-vertex model the "momentum" and the "phase shifts" are given by

$$\mathrm{e}^{ip(q)} = f(-q,\gamma), \qquad \mathrm{e}^{i\phi(q)} = f(q,2\gamma), \tag{2.6}$$

$$f(q,a) = \frac{\mathrm{sh}\frac{1}{2}(q+ia)}{\mathrm{sh}\frac{1}{2}(q-ia)}. \tag{2.7}$$

There exist detailed investigations of the Bethe ansatz equations (2.5) (see e.g. ref. [14]). The analysis simplifies for the discrete values of the "coupling"

$$\gamma = \pi/\nu, \qquad \nu = 2,3,4,\ldots. \tag{2.8}$$

We are interested in these values. So we assume eq. (2.8) in the following. For $M \gg 1$ the Bethe ansatz equations have solutions of string type:

(i) k^+ string

$$q = q^{(k)} + i\gamma(-k+1,-k+3,\ldots,k-1), \qquad k=1,2,\ldots,\nu-1. \tag{2.9}$$

(ii) 1^- string

$$q = i\pi + q^{(a)}. \tag{2.10}$$

The k^+ and 1^- strings may be interpreted as "bound states" of k pseudoparticles and as an "antipseudoparticle", respectively. The Bethe ansatz equations (18) can be rewritten in terms of the real "rapidities" $q^{(k)}$ and $q^{(a)}$ of these states

$$e^{iMP_k(q_j^{(k)})}\prod_{l=1}^{\nu-1,a}\prod_{i=1}^{m_l}e^{i\phi_{kl}(q_j^{(k)}-q_i^{(l)})}=-1, \tag{2.11}$$

where the momenta p_k are given by

$$e^{ip_k(q)}=\prod_{r=1}^{k}f(-q-i\gamma(2r-k-1),\gamma)=f(-q,k\gamma);\qquad k=1,2,\dots,\nu-1,$$

$$e^{ip_a(q)}=f(-q-i\pi,\gamma)=-f(-q,\pi+\gamma), \tag{2.12}$$

and the phase shifts ϕ_{kl} by

$$e^{i\phi_{kl}(q)}=f(q,(k+l)\gamma)f^2(q,(k+l-2)\gamma)\cdots f^2(q,(|k-l|+2)\gamma)f(q,|k-l|\gamma),$$

$$e^{i\phi_{ka}(q)}=f(q_1(k+1)\gamma+\pi)f(q_1(k-1)\gamma+\pi),$$

$$e^{i\phi_{aa}(q)}=e^{i\phi(q)}. \tag{2.13}$$

Taking the logarithm of eq. (2.11)

$$2\pi I_j^{(k)}=Mp_k\left(q_j^{(k)}\right)+\sum_{l,i}\phi_{kl}\left(q_j^{(k)}-q_i^{(l)}\right), \tag{2.14}$$

we observe that the roots $q=\{q_j^{(k)}\}$ are given by sets of numbers $\{I^{(k)}\}\subset\mathbb{Z}$, or $\mathbb{Z}+\frac{1}{2}$. For the "ground state" (corresponding to the maximal eigenvalue $\lambda^{(q)}$) there are only real roots ($k=1$) without holes, which means that the numbers $I_j^{(l)}$ accept all possible values in an interval $I_{\min}^{(1)}\leqslant I_j^{(1)}\leqslant I_{\max}^{(1)}$. For "excited states" there may be "holes" in the distribution of the $I_j^{(l)}$ and higher strings ($k>1$) may appear. For large lattices $M\gg 1$ and many roots $m\gg 1$ it is natural to introduce densities of k-strings $\rho_k(q)$ defined by

$$2\pi\frac{dI^{(k)}}{dq}=\sigma_k\left[\rho_k(q)+\hat{\rho}_k(q)\right];\qquad k=1,\dots,\nu-1,a, \tag{2.15}$$

where $\hat{\rho}_k(q)$ are the densities of holes and $\sigma_k=\operatorname{sgn}p_k'(q)=1$ for $k=1,\dots,\nu-1$ and -1 for $k=a$.

Taking the derivative d/dq of eq. (2.14) the Bethe ansatz equations can be written as a system of integral equations

$$\sigma_k(\rho_k+\hat{\rho}_k)=Mp_k'+\phi_{kl}'*\rho_l, \tag{2.16}$$

where the convolution is defined by $(f*g)(q)=\int(dq'/2\pi)f(q-q')g(q')$. The eigenvalues of the transfer matrix eq. (2.4) are given by

$$\lambda_A(\theta,\gamma) = \exp\left\{ -i\sum_k \int \frac{\mathrm{d}q}{2\pi} \rho_k(q) [p_k(q+2i\theta) - \pi_k] \right\}, \tag{2.17}$$

where $\pi_k = \pi$ for k odd or $k = a$ and $\pi_k = 0$ for k even.

For the ground state in the limit $M \to \infty$ we obtain from eq. (2.16) by means of Fourier transformation the root density

$$\rho_k(q) = \delta_{k1} \frac{M}{2} \frac{\nu}{\cosh(\nu q/2)}. \tag{2.18}$$

Since in the region $0 < \theta < \frac{1}{2}\gamma$

$$|\lambda_A| > |\lambda_D| \tag{2.19}$$

the partition function in the thermodynamic limit $M, N \to \infty$ is

$$Z_{M,N} \simeq \lambda^N_{A,\max} = \mathrm{e}^{-MNf_\infty},$$

where the "free energy" per site follows from eqs. (2.17), (2.18)

$$f_\infty = \int_0^\infty \frac{\mathrm{d}x}{x} \frac{\mathrm{sh}(2\theta x)\mathrm{sh}[x(\pi-\gamma)]}{\mathrm{ch}(\gamma x)\mathrm{sh}(\pi x)} = \log S^{\mathrm{SG}}(i\pi\theta/\gamma) \tag{2.20}$$

where S^{SG} is the sine-Gordon soliton-soliton S-matrix.

3. Six-vertex model – finite size corrections (method (a))

Since the six-vertex model is a critical eight-vertex model we expect in the scaling limit conformal invariance. We shall calculate the conformal anomaly number or the central charge c by method (a) explained in the introduction.

The partition function for a two dimensional classical statistical model (c.f. eq. (1.10)) on a strip of width N with periodic boundary conditions

$$Z = \mathrm{tr}[\tau^N] = \mathrm{tr}\,\mathrm{e}^{-H/T} \tag{3.1}$$

can be reinterpreted as the partition function of a one-dimensional quantum statistical model with finite temperature $T = 1/N$, where the "hamiltonian" is given by

$$H = -\ln \tau. \tag{3.2}$$

The eigenvalues of the six-vertex model are real for the isotropic case $\theta = \frac{1}{2}\gamma$ (c.f. eq. (2.4)) which means that H is hermitian. Because of technical reasons we consider first $0<\theta<\frac{1}{2}\gamma$. The value $\theta=\frac{1}{2}\gamma$ is obtained as a limiting case. We introduce the auxiliary system

$$Z(\mu)=\mathrm{Tr}\,\mathrm{e}^{-H(\mu)/T}, \qquad T=1/N, \tag{3.3}$$

where the "hamiltonian"

$$H(\mu)=H(0)+\mu\phi \tag{3.4}$$

is defined by its eigenvalues (see eq. (2.17)).

$$\begin{aligned} E(\mu) &= -\mathrm{Re}\log\lambda_A-\mu\,\mathrm{Im}\log\lambda_A \\ &= \sum_k\int\frac{\mathrm{d}q}{2\pi}\rho_k(q)e_k(q), \\ e_k(q) &= \mathrm{Re}\,ip_k(q+2i\theta)+\mu\,\mathrm{Im}\,i\big(p_k(q+2i\theta)-\pi_k\big). \end{aligned} \tag{3.5}$$

Because of $|\lambda_A|>|\lambda_D|$ for $0<\theta<\frac{1}{2}\gamma$ (cf. eq. (2.19)) we have from eqs. (1.10), (2.4)

$$Z=\sum_q\big[\lambda_A^{(q)}+\lambda_D^{(q)}\big]=Z(\mu=i)\big[1+\mathrm{O}(\mathrm{e}^{-\delta N})\big]. \tag{3.6}$$

We consider first $Z(\mu)$ for real μ ($|\mu|<\mathrm{tg}(\nu\theta)$) and take finally the analytic continuation $\mu\to i$ having in mind the identity $Z_\mu=Z_0\langle\exp(N\mu\phi)\rangle$.

We borrow now from thermodynamics the method to calculate the free energy of a statistical system for finite temperature

$$F=-T\ln\mathrm{tr}\,\mathrm{e}^{-H(\mu)/T}=E-TS \qquad \text{with}\ \delta(F)=0, \tag{3.7}$$

where S is the entropy. Following Yang and Yang [15] we write

$$S=\sum_k\int\frac{\mathrm{d}q}{2\pi}\big[(\rho_k+\hat\rho_k)\ln(\rho_k+\hat\rho_k)-\rho_k\ln\rho_k-\hat\rho_k\ln\hat\rho_k\big]. \tag{3.8}$$

Minimizing the free energy using the Bethe ansatz equations (2.16) we obtain another set of integral equations for the densities of strings ρ_k and holes $\hat\rho_k$

$$Ne_k-\ln(1+\hat\rho_k/\rho_k)+(1-\phi')_{kl}*\sigma_l\log(1+\rho_l/\hat\rho_l)=0 \tag{3.9}$$

Inverting the Bethe ansatz equations (2.16) we obtain

$$\begin{aligned} &\rho_k+\hat\rho_k=\delta_{k1}M_c+c*(\hat\rho_{k-1}+\hat\rho_{k+1})+c*\rho_a\delta_{k,\nu-2}, \\ &\qquad\qquad k=1,2,\dots,\nu-1 \\ &\rho_a+\hat\rho_a=c*\hat\rho_{\nu-2} \end{aligned} \tag{3.10}$$

and correspondingly we derive from eq. (3.9)

$$\chi_k + \hat{\chi}_k = -\delta_{k1} N d' + c * (\hat{\chi}_{k-1} + \hat{\chi}_{k+1}) + c * \chi_a \delta_{k,\nu-2}, \qquad k = 1, \dots, \nu - 1,$$

$$\chi_a + \hat{\chi}_a = -c * \hat{\chi}_{\nu-2}, \tag{3.11}$$

where $\chi_k = [\ln(1+\rho_k/\hat{\rho}_k)]'$, $\hat{\chi}_k = \chi_k \hat{\rho}_k/\rho_k$ and

$$c(q) = \frac{1}{2} \frac{\nu}{\cosh(\frac{1}{2}\nu q)}, \tag{3.12}$$

$$d'(q) = \tfrac{1}{2} i [c(q+2i\theta) - c(q-2i\theta)] + \tfrac{1}{2}\mu [c(q+2i\theta) + c(q-2i\theta)],$$

$$p'_k = (1-\phi')_{k1} * c, \qquad e_k = (1-\phi')_{k1} * d. \tag{3.13}$$

has been used. Since the low temperature (large N) corrections to the sum in eq. (1.13a) are due to low energy excitations ($q \to \pm\infty$) it is sufficient to solve eqs. (3.10) and (3.11) asymptotically. Similar equations have been solved for Heisenberg models [16]. Since $c(q)$ and $d(q)$ decay exponentially $c(q) \approx \nu e^{\mp \nu q/2}$ and $d'(q) \approx c(q)\,(\pm \sin\nu\theta + \mu\cos\nu\theta)$ for $q \to \pm\infty$ one finds from eqs. (3.10) and (3.11)

$$\chi_k(q) = -\frac{N}{M} \sigma_k (\pm \sin\nu\theta + \mu\cos\nu\theta) \rho_k(q). \tag{3.14}$$

In addition, from eq. (3.11) follows that the functions

$$y_k^{\pm}(q) \equiv \hat{\rho}_k(q')/\rho_k(q') = \hat{\chi}_k(q')/\chi_k(q'),$$

$$q' \equiv q_{\pm}(q, N) = q \pm \frac{2}{\nu}[\log 2N(\sin\nu\theta \pm \cos\nu\theta)] \tag{3.15}$$

are independent of N for large N since they fulfill the system of integral equations

$$\ln y_k^{\pm} = -\delta_{k1} e^{\mp \nu q/2} + c * \ln[(1+y_{k-1}^{\pm})(1+y_{k+1}^{\pm})] + \delta_{k,\nu-2} c * \log[1 + 1/y_a^{\pm}],$$

$$\log y_a^{\pm} = -c * \ln(1 + y_{\nu-2}^{\pm}). \tag{3.16}$$

The low temperature (large N) corrections to the entropy $S = S_+ + S_- + O(N^{-2})$

(note that $S(N=\infty)=0$) are given by the integrals eq. (3.8) at $q\to\pm\infty$

$$S_+=\sum_k\int_{Q\gg 1}^{\infty}\frac{\mathrm{d}q}{2\pi}\left[\rho_k(q)\ln(1+\hat{\rho}_k(q)/\rho_k(q))\right.$$
$$\left.+\hat{\rho}_k(q)\ln(1+\rho_k(q)/\hat{\rho}_k(q))\right] \tag{3.17}$$

and correspondingly for S_-. We substitute

$$x_k(q)=1/(1+y_k^+(q)) \tag{3.18}$$

which implies using eqs. (3.14) and (3.15)

$$S_+=\frac{M}{N}\frac{1}{2\pi}\frac{1}{\sin\nu\theta+\mu\cos\nu\theta}\sum_k\int_{x_k(q_+^{-1}(Q,N))}^{x_k(\infty)}\mathrm{d}x\left[\frac{\ln x}{1-x}+\frac{\ln(1-x)}{x}\right]. \tag{3.19}$$

For large N, $x_k(q_+^{-1}(Q,N))$ can be replaced by $x_k(-\infty)$. The integration bounds $x_k(\pm\infty)$ can be obtained from the asymptotic solutions of eq. (3.16)

$$x_k(\infty)=1/(k+1)^2,\qquad x_k(-\infty)=1/k^2,\qquad k=1,\dots,\nu-2,$$
$$x_{\nu-1}(\infty)=1-x_a(\infty)=1/\nu,\qquad x_{\nu-1}(\infty)=1-x_a(\infty)=1/(\nu-1). \tag{3.20}$$

Inserting these values in eq. (3.19) we obtain for the sum

$$\left[\sum_{k=1}^{\nu-2}\int_{1/k^2}^{1/(k+1)^2}+2\int_{1/(\nu-1)}^{1/\nu}\right]\mathrm{d}x\left[\frac{\ln x}{1-x}+\frac{\ln(1-x)}{x}\right]=\frac{\pi^2}{3} \tag{3.21}$$

independent of ν. Together with analogous results for S_- we obtain for the entropy for large N

$$S=\frac{M}{N}\frac{\pi}{3}\frac{\sin\nu\theta}{\sin^2\nu\theta-\mu^2\cos^2\nu\theta}+\mathrm{O}(N^{-2}). \tag{3.22}$$

The large N correction for the free energy at $\mu=i$ (cf. eq. (3.6)) follow from eq. (3.7) and $S=-\partial F/\partial T$

$$NF=-\ln Z\Big|_{\mu=i}=N\left[F_\infty-\frac{1}{2N^2}\frac{\partial S}{\partial(1/N)}+\mathrm{O}(N^{-3})\right]$$
$$=NF_\infty-\frac{M}{N}\frac{\pi}{6}\sin\nu\theta+\mathrm{O}(N^{-2}). \tag{3.23}$$

For $\theta=\frac{1}{2}\gamma=\pi/(2\nu)$ the vertex weights $\omega_1=\omega_2$ and $\omega_3=\omega_4$ are equal (cf., eq. (2.2)). So we expect the model to be isotropic and conformal invariant in the scaling limit. Comparing eq. (3.23) for $\theta = \frac{1}{2}\gamma$ with the finite size correction formulas (1.13a, b) we conclude for the six-vertex model (in the scaling limit) the central charge of the Virasoro algebra

$$c^{\text{6-vertex}} = 1\,. \tag{3.24}$$

For general θ, $0 < \theta < \gamma$ the model is not isotropic. But the lattice may be deformed in order to recover rotational symmetry. The resulting volume renormalization cancels the factor $\sin \nu\theta$ in eq. (3.23) such that eq. (3.24) also hold for the general six-vertex model.

4. The critical q-state Potts model

The q-state Potts model is defined by the partition function

$$Z^{\text{Potts}}(\beta) = \sum_{\{\sigma\}} \exp\Big(\beta \sum_{\langle xy\rangle} \delta_{\sigma(x),\,\sigma(y)}\Big), \qquad 1 \leqslant \sigma \leqslant q\,. \tag{4.1}$$

It was shown [17] to be at the critical point $\beta_c = \ln(\sqrt{q} + 1)$ equivalent to a six-vertex model with

$$q = 4\cos^2\gamma\,, \qquad \gamma = \pi/\nu\,. \tag{4.2}$$

Baxter, Kelland, and Wu [18] showed that for cylindric boundary conditions one has to introduce a "seam" from the bottom to the top of the cylinder which has to carry extra weights. This can be written in terms of the six-vertex transfer matrix as

$$Z^{\text{Potts}}(\beta_c) = \text{tr}[\tau^N e^{2\pi i S_3/\nu}]\,, \tag{4.3}$$

where S_3 is the "total spin" counting the number of "up" minus "down" arrows on the seam

$$S_3 = \sum_{i=1}^{M} \tfrac{1}{2}\big(1 \otimes \cdots \otimes \sigma_3^{(i)} \otimes \cdots \otimes 1\big)\,. \tag{4.4}$$

Eq. (4.3) means we have to calculate

$$Z_{MN}^{\text{Potts}} = Z_{MN}^{\text{6-vertex}}\langle \exp(2\pi i S_3/\nu)\rangle\,, \tag{4.5}$$

where $\langle\,\cdot\,\rangle$ may be understood in terms of the quantum partition function (3.1). We calculate the expectation value in eq. (4.5) introducing a "magnetic field" by the following substitution in eq. (3.3)

$$H(\mu) \to H(\mu) - S_3 T h\,, \qquad T = 1/N\,. \tag{4.6}$$

The eigenvalue s_3 of S_3 corresponding to a state given by eq. (2.16)

$$s_3 = \tfrac{1}{2}M - m = \tfrac{1}{2}M - \int \frac{dq}{2\pi}\left[\sum_{k=1}^{\nu-1} k\rho_k(q) + \rho_a(q)\right]$$

$$= \frac{\nu}{2}\int \frac{dq}{2\pi}\left[\hat{\rho}_{\nu-1}(q) - \rho_a(q)\right]. \tag{4.7}$$

The last equality follows from the Bethe ansatz equations (2.16). The low temperature (large N) expectation value of S_3 for the quantum statistical problem (3.1) with magnetic field is analogously to eq. (3.17) is given by

$$\langle S_3\rangle^+ = \frac{\nu}{2}\int_{Q\gg 1}^{\infty} \frac{df}{2\pi}\left[\hat{\rho}_{\nu-1}(q) - \rho_a(q)\right] \tag{4.8}$$

due to large $q \to \infty$ excitations (and correspondingly for $q \to -\infty$). The computation of $\langle S_3\rangle$ is similar to that one for Heisenberg models in ref. [16]. From eqs. (3.14) and (3.15) one obtains

$$\langle S_3\rangle^+ = \frac{\nu}{2}\frac{M}{N}\frac{1}{\sin\nu\theta + \mu\cos\nu\theta}\int_Q^{\infty}\frac{dq}{2\pi}\left[-\chi_{\nu-1}(q) - \chi_a(q)\right]$$

$$= \frac{\nu}{2}\frac{M}{N}\frac{1}{\sin\nu\theta + \mu\cos\nu\theta}\ln\frac{1+y_{\nu-1}(\infty)}{1+y_{\nu-1}(-\infty)}\,\frac{1+y_a^{-1}(\infty)}{1+y_a^{-1}(-\infty)} + O(N^{-2}). \tag{4.9}$$

The values $y_k(\pm\infty)$ can be obtained analogously to eqs. (3.20) with the extra terms in eq. (3.16) $h\nu\delta_{k_1(\nu-1)}/2$ and $\frac{1}{2}h\nu$ on the r.h.s., respectively. The result is

$$y_k(\infty) = \frac{\mathrm{sh}^2\left(\frac{1}{2}h(k+1)\right)}{\mathrm{sh}^2\left(\frac{1}{2}h\right)} - 1, \qquad y_k(-\infty) = \frac{\mathrm{sh}^2\left(\frac{1}{2}h\nu k/(\nu-1)\right)}{\mathrm{sh}^2\left(\frac{1}{2}h\nu/(\nu-1)\right)};$$

$$k = 1, \dots, \nu-2,$$

$$y_{\nu-1}(\infty) = \frac{e^{h\nu}}{y_a(\infty)} = e^{-h\nu/2}\frac{\mathrm{sh}\frac{1}{2}h(\nu-1)}{\mathrm{sh}\frac{1}{2}h},$$

$$y_{\nu-1}(-\infty) = \frac{e^{h\nu}}{y_a(-\infty)} = \frac{\mathrm{sh}\left(\frac{1}{2}h\nu(\nu-2)/(\nu-1)\right)}{\mathrm{sh}\left(\frac{1}{2}h\nu/(\nu-1)\right)}. \tag{4.10}$$

Inserting these values in eq. (4.9) and the corresponding ones for $\langle S_3\rangle^-$ one obtains (for $\mu = i$) the average "magnetization"

$$\langle S_3 \rangle = \frac{M}{N} \frac{h\nu}{2\pi(\nu-1)} \sin \nu\theta + O(N^{-2}). \tag{4.11}$$

This implies for any α

$$\langle \exp(\alpha S_3) \rangle_{h=0} \simeq \exp\left[\frac{\alpha^2}{2} \frac{M}{N} \frac{\nu}{\nu-1} \frac{\sin \nu\theta}{2\pi}\right] \quad \text{for } N \to \infty. \tag{4.12}$$

Comparing eqs. (1.13), (4.5), (4.12) we conclude that the conformal invariant q-state Potts model in the scaling limit correspond to a representation of the Virasoro algebra with central charge

$$c^{\text{Potts}} = 1 - \frac{6}{\nu(\nu-1)}, \tag{4.13}$$

where $q = 4\cos^2(\pi/\nu)$.

5. Six-vertex model – finite size corrections (method (b))

We compute in this section the finite size correction to the largest eigenvalue of the six-vertex model transfer matrix. This provides the dominant finite size corrections to the free energy. We do that for periodic boundary conditions in the gapless regime. In the antiferromagnetic domain with non-zero gap these corrections were calculated in ref. [19] and they turned out to be exponentially small in N.

It is useful to consider the function [19]

$$t_N(q) = p(q) + \frac{1}{N} \sum_{i=1}^{m} \phi(q - q_i). \tag{5.1}$$

This function is continuous and monotonically increasing for real q. At the real roots of the BAE (2.14) where M is now replaced by N,

$$t_N\left(q_j^{(k)}\right) = \frac{2\pi}{N} I_j^{(k)}. \tag{5.2}$$

We define the function

$$\sigma_N(q) = \frac{1}{2\pi} \frac{\mathrm{d}t_N}{\mathrm{d}q}. \tag{5.3}$$

We can limit ourselves here to consider only real roots q_j since the complex roots and holes do not contribute to the dominant finite size corrections as will be discussed later on. In this case $\sigma_N(q)$ tends to the density of real roots $\rho_1(q)$ for $N \to \infty$

$$\sigma_\infty(q) = \lim_{N\to\infty} \sigma_N(q) = \frac{\rho_1(q)}{N} \tag{5.4}$$

as it follows from eq. (5.2), (5.3). The difference between eqs. (5.1), (5.2) for $k=j$ and $j+1$ yields in the $N=\infty$ limit an integral equations for $\sigma_\infty(q)$

$$\sigma_\infty(q) = \frac{1}{2\pi}p'(q) + \int_{-\infty}^{+\infty} \frac{\mathrm{d}q'}{2\pi}\phi'(q-q')\sigma_\infty(q'), \tag{5.5}$$

with the solution already given in (2.18)

$$\sigma_\infty(q) = \int_0^\infty \frac{\mathrm{d}k}{4\pi}\frac{\cos kq}{\mathrm{ch}\, k\gamma} = \frac{1}{4\gamma\cosh(\pi q/2\gamma)}. \tag{5.6}$$

Now, it is possible to compute λ_A and λ_D in the $N=\infty$ limit. The free energy will be given by the largest number between them. Since $|\lambda_A| > |\lambda_D|$ for $\frac{1}{2}\gamma > \theta > 0$

$$F_\infty = -\lim_{N\to\infty}\frac{1}{N}\log\lambda_A(\theta,\gamma). \tag{5.7}$$

So, the term λ_D gives exponentially small contributions to F_N. Since, we are interested in the power corrections to F_∞ we concentrate on λ_A. Higher eigenvalues of $\tau(\theta)$ give subdominant corrections for large N.

Define the function

$$L_N(\theta) = -\frac{1}{N}\log\lambda_A(\theta,\gamma) + \lim_{N\to\infty}\frac{1}{N}\log\lambda_A(\theta,\gamma). \tag{5.8}$$

It can be reexpressed as follows with the help of eq. (2.4)

$$L_N(\theta) = -i\int_{-\infty}^{+\infty}\mathrm{d}q\,p(q+2i\theta)\left\{\frac{1}{N}\sum_{j=1}^{m}\delta(q-q_j) - \sigma_N(q)\right\}$$
$$+ i\int_{-\infty}^{+\infty}\mathrm{d}q\,p(q+2i\theta)\left[\sigma_N(q)-\sigma_\infty(q)\right]. \tag{5.9}$$

As in ref. [19], $\sigma_N(q) - \sigma_\infty(q)$ can be expressed as

$$\sigma_N(q) - \sigma_\infty(q) = \int_{-\infty}^{+\infty}\mathrm{d}q'\,J(q-q')\left\{\frac{1}{N}\sum_{j=1}^{m}\delta(q'-q_j) - \sigma_N(q')\right\}. \tag{5.10}$$

Here the function $J(q)$ reads

$$J(q) = -\int_0^{\infty} \frac{\mathrm{d}x}{2\pi} \frac{\cos(qx)\mathrm{sh}(\pi-2\gamma)x}{\mathrm{sh}[(\pi-\gamma)x]\cosh(\gamma x)}. \tag{5.11}$$

Its properties are summarized in the appendix.

Inserting eq. (4.10) in eq. (5.9) and using eq. (A.5) yields

$$L_N(\theta) = -i\int_{-\infty}^{+\infty} \mathrm{d}q \left[\frac{\pi\gamma}{2(\pi-\gamma)} + 2\,\mathrm{arctg}\,\mathrm{e}^{\pi(q/2-i\theta)/\gamma}\right] \times\left\{\frac{1}{N}\sum_{j=1}^{m}\delta(q-q_j) - \sigma_N(q)\right\}. \tag{5.12}$$

Eq. (5.12) has the appropriate structure in order to study its large N behaviour following the method used in ref. [19] for models with non-zero gap. As in ref. [19] any expression of the form

$$I_N = \int_{-\infty}^{+\infty} \mathrm{d}q f(q)\left\{\frac{1}{N}\sum_{j=1}^{m}\delta(q-q_j) - \sigma_N(q)\right\} \tag{5.13}$$

has a Fourier expansion

$$I_N = \sum_{\substack{\alpha=-\infty \\ \alpha\neq 0}}^{+\infty} (-1)^{\alpha} T_{N\alpha}, \tag{5.14}$$

where

$$T_n \equiv \int_{-\infty}^{+\infty} f(q)\sigma_N(q)\mathrm{e}^{int_N(q)}\,\mathrm{d}q. \tag{5.15}$$

In the present case one cannot use the saddle point method as in ref. [19] to evaluate T_n for large n. One can still approximate σ_N and t_N by σ_∞ and t_∞ where σ_∞ is given by eq. (5.6) and

$$t_\infty(q) = 2\,\mathrm{arctg}(\mathrm{e}^{\pi q/2\gamma}). \tag{5.16}$$

So, we set

$$T_n \simeq T_n^{\mathrm{as}} \equiv \int_{-\infty}^{+\infty} \mathrm{d}q f(q)\sigma_\infty(q)\mathrm{e}^{int_\infty(q)},$$

or

$$T_n^{\mathrm{as}} = \int_0^{\pi} \frac{\mathrm{d}t}{2\pi}\mathrm{e}^{int} f(q(t)), \tag{5.17}$$

where

$$q(t)=\frac{2\gamma}{\pi}\log\left(\operatorname{tg}\tfrac{1}{2}t\right). \tag{5.18}$$

In eq. (5.14) we need $T_{N\alpha}$ where N is a large even number. In this case $T_{N\alpha}\simeq T_{N\alpha}^{\mathrm{as}}$ is dominated by the end points of the integral (5.17), namely $t=0$ and π provided $f(q(t))$ is smooth. For a typical exponentially decreasing function

$$f(q)\underset{q\to\pm\infty}{=}f_{\pm}+g_{\pm}\mathrm{e}^{-a|q|}+\text{smaller terms} \tag{5.19}$$

one finds from eq. (5.17)

$$T_n^{\mathrm{as}}\underset{n\gg 1}{=}\frac{f_+-f_-}{2\pi ni}+\frac{\Gamma(1+2a\gamma/\pi)}{i\pi}\,\frac{g_+\mathrm{e}^{-ia\gamma}-g_-\mathrm{e}^{ia\gamma}}{(2n)^{1+2a\gamma/n}}+\text{higher order}. \tag{5.20}$$

The first term vanishes upon summation on α (eq. (5.14)) and the second term yields

$$I_N=\frac{\Gamma(1+2a\gamma/\pi)}{i\pi(2N)^{1+2a\gamma/\pi}}\left(q_+\mathrm{e}^{-ia\gamma}-q_-\mathrm{e}^{ia\gamma}\right)\zeta\left(1+\frac{2\alpha\gamma}{\pi}\right)\left(2^{-2a\gamma/\pi}-1\right)$$
$$\times\left(1-\mathrm{e}^{-2ai\gamma}\right)+\text{higher orders in }1/N. \tag{5.21}$$

Here $\zeta(z)$ is the Riemann zeta function.

Let us apply this method to $L_N(\theta)$ (eq. (5.12)). One has

$$f(q)=-i\left[\frac{\pi\gamma}{2(\pi-\gamma)}+2\operatorname{arctg}\mathrm{e}^{\pi(q/2-i\theta)/\gamma}\right]. \tag{5.22}$$

So,

$$f_-=-\frac{i\pi\gamma}{2(\pi-\gamma)},\qquad f_+=-\frac{i\pi(2\pi-\gamma)}{2(\pi-\gamma)},\qquad g_{\pm}=\pm 2i\mathrm{e}^{\pm i\pi\theta/\gamma}$$

and $a=\pi/2\gamma$. Eq. (5.21) simplifies considerably yielding for this case

$$L_N(\theta)=-\frac{\pi}{6N^2}\sin\frac{\pi\theta}{\gamma}+\mathrm{O}\left(\frac{1}{N^3}\right). \tag{5.23}$$

This exactly coincides with the expression obtained by method (a) (eq. (3.23)). Let us now analyze it in the light of conformal invariant theories.

Unless $\theta=\frac{1}{2}\gamma$ the six-vertex model is not rotationally invariant as one sees from the values of the vertex weights. Moreover, for long distances one expects to find rotational invariance after length rescaling in one direction, since the model is in its critical regime. This can be investigated studying the excitation spectrum. The eigenvalue for $\tau(\theta)$ reads for a hole located at $q=\varphi$

$$\lambda(\theta,\varphi) = -2i \operatorname{arctg} e^{\pi(\varphi/2 - i\theta)/\gamma}. \tag{5.24}$$

Since the momentum operator reads here

$$\mathscr{P} = i \log \tau(0) \tag{5.25}$$

the momentum of the hole equals to

$$p = 2 \operatorname{arctg} e^{\pi\varphi/2\gamma}. \tag{5.26}$$

In this context the "hamiltonian" is

$$H = -\operatorname{Re} \log \tau(\theta). \tag{5.27}$$

So, for low "energy", $\varphi \to -\infty$, we have

$$p = 2 e^{\pi\varphi/2\gamma} + O(e^{\pi\varphi/\gamma}),$$

$$\varepsilon = p \sin\frac{\pi\theta}{\gamma} + O(p^2).$$

Hence, we have to redefine H by a factor $(\operatorname{cosec}(\pi\theta/\gamma))$ in order to have an ultrarelativistic dispersion law and then rotational invariance. After this finite renormalization

$$\tilde{L}_N(\theta) = \frac{1}{\sin(\pi\theta/\gamma)} L_N(\theta) = \frac{\pi}{6N^2} + O\left(\frac{1}{N^3}\right) \tag{5.28}$$

and we find $c=1$ for the six-vertex model for all values of θ in the gapless regime.

Let us now consider excited states of the six-vertex model in the critical regime. Finite size corrections to their energies give the conformal dimensions of the operators associated to them [12].

Let us consider a state where $m = \frac{1}{2}N - 1$. For large N one can build a state like that, assuming that the roots of the BAE are real and restricted to a finite but large interval $(-b, +b)$ [20]. In this way one can take $N = \infty$ and finds for the density of roots [20]

$$\sigma(q, y) = \frac{1}{2\pi} p'(q) + \int_{-b}^{b} \frac{dr}{2\pi} \sigma(r, y) \phi'(q - r), \tag{5.29}$$

where y is the magnetization

$$y = 1 - \frac{2m}{N}$$

and $b = b(y)$. For $m = \frac{1}{2}N - 1$ one finds $y = 2/N \ll 1$, so eq. (5.29) can be solved approximately using Wiener-Hopf techniques. Define

$$L_y(\theta) = -\lim_{N\to\infty} \frac{1}{N}\left[\log \Lambda_+(\theta, y) - \log \Lambda_+(\theta,0)\right]. \tag{5.30}$$

Subtracting eq. (5.29) from eq. (5.5) one derives that

$$\sigma(q, y) - \sigma(q,0) = \int_{b(y)<|\mu|<\infty} \mathrm{d}q' J(q-q')\sigma(q', y) \tag{5.31}$$

where $\sigma(q,0) = \sigma_\infty(q)$. Moreover, it can be shown from eqs. (5.29)–(5.31) that

$$L_y(\theta) = \tfrac{1}{2}i\gamma y + i\int_{|q|>b(y)} \mathrm{d}q\, \sigma(q, y) t_\infty(q - 2i\theta), \tag{5.32}$$

where eq. (A.5) was used and $t_\infty(q)$ is given by eq. (5.16). Using the perturbative solution of eq. (5.21) given in ref. [20] one finds from eq. (5.32)

$$L_y(\theta) = \tfrac{1}{4}y^2(\pi-\gamma)\sin\frac{\pi\theta}{\gamma}\left[1 + \mathrm{O}(y^2) + \mathrm{O}(y^{4\gamma/(\pi-\gamma)})\right]. \tag{5.33}$$

In the calculation we drop a term $-\frac{1}{2}iy\pi$ since this represents a contribution to the partition function equal to one. Now, setting $y = 2/N$ and renormalizing $L_y(\theta)$ according to eq. (5.28) yields

$$\tilde{L}_y(\theta) = \frac{\pi-\gamma}{N^2} + \mathrm{O}\left(\frac{1}{N^4}, \frac{1}{N^{2+4\gamma/(\pi-\gamma)}}\right). \tag{5.34}$$

According to conformal field theory arguments (eq. (1.14)) this corresponds to an operator of conformal dimension

$$x = \tfrac{1}{2}\left(1 - \frac{\gamma}{\pi}\right). \tag{5.35}$$

It can be identified with an "electric field" [21].

Let us finally indicate how the properties for the XXZ Heisenberg chain follow from the previous results. The XXZ hamiltonian is related to the transfer matrix of the six-vertex model by

$$H_{\mathrm{XXZ}} = -\sin\gamma \frac{\partial}{\partial\theta}\log \tau(\theta)\bigg|_{\theta=0}. \tag{5.36}$$

So one finds for the ground state energy of the XXZ chain from eqs. (5.8), (5.23) and (5.30) [22,23]

$$E_N - E_\infty = -\frac{\sin\gamma}{\gamma}\frac{\pi^2}{6N^2} + \text{higher orders}. \tag{5.37}$$

In the same way the expression (5.35) for the conformal dimension of an electric operator holds in the XXZ model [23].

Appendix

The function $J(q)$ is an even meromorphic function of q defined by eq. (5.11) (see also ref. [19] and [20]). It fulfils the properties

$$J(q) = J(-q), \tag{A.1}$$

$$J(q - i\gamma) + J(q + i\gamma) = \frac{1}{2\pi i}\frac{\partial}{\partial q}\log f\left(\frac{q}{1-\gamma/\pi}, \frac{\gamma}{1-\gamma/\pi}\right) \tag{A.2}$$

$$J(q)_{|q|\to\infty} = \frac{1}{1-\gamma/\pi}\operatorname{tg}\left(\frac{\gamma}{1-\gamma/\pi}\right)\frac{e^{-|q|/(1-\gamma/\pi)}}{2\pi} + \frac{1}{2\gamma}\operatorname{tg}\left(\frac{\pi^2}{2\gamma}\right)e^{-\pi|q|/2\gamma} + \text{smaller terms}. \tag{A.3}$$

This last equation holds when $\pi/(\pi-\gamma)$ is not a rational number.

$$\int_{-\infty}^{+\infty} dq J(q) = -\frac{1}{2}\frac{\pi - 2\gamma}{\pi - \gamma}, \tag{A.4}$$

$$\int_{-\infty}^{+\infty} d\mu\, J(\lambda-\mu)p(\mu) + p(\lambda) = \frac{\gamma}{2(1-\gamma/\pi)} + 2\operatorname{arctg}(e^{\pi\lambda/2\gamma}). \tag{A.5}$$

When $\gamma = \frac{1}{3}\pi$

$$J(q) = -\frac{q}{2\pi^2}\frac{q}{\operatorname{sh}\frac{3}{2}q}.$$

$J(q)$ is analytic for $|\operatorname{Im} q| < 2\gamma$.

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