The Bootstrap program for 1+1 dimensional field theoretic models with soliton behaviour

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Abstract

A review is given of the present status of the bootstrap program for 1+1 dimensional field theoretic models with soliton behaviour. From the existence of infinitely many conservation laws, unitarity, crossing, "minimality", internal symmetry, and some assumptions on the particle spectrum the exact S-matrix is derived explicitly. We repeat the procedure for the massive Thirring-model (alias Sine-Gordon) S-matrix, starting with two fermion (alias soliton) scattering and then derive the bound state (alias breather) S-matrix. The same complete S-matrix is obtained by starting with the determination of the boson (breather) S-matrix and then deriving the boson-soliton and soliton-soliton scattering from the assumption that the soliton is a soliton-boson bound state. By means of the same procedure we obtain the Gross-Neveu S-matrix including the kink scattering. Then for some models generalized form factors and Green's functions for the $Z(2)$-Ising model in the scaling limit are calculated.

1 The S-matrix

For some field theoretic models we calculate exactly S-matrix elements [1] like

$$^{out}\langle\, p_1', \ldots, p_n' \,|\, p_1, \ldots, p_n \,\rangle^{in}$$

for the scattering of any number and kinds of particles appearing in the models.

1.1 Introduction: The models

i) The Sine-Gordon (SG) model defined by the relativistic wave equation

$$\Box\, \phi + \frac{\alpha}{\beta} \sin \beta\phi = 0 \tag{1}$$

is on the classical level completely integrable by means of the inverse scattering method [2]. There are solutions called soliton (antisoliton)

$$\phi_{sol}(x,t) = \phi_{antisol}(-x,t) = \frac{4}{\beta} \tan^{-1} e^{\sqrt{\alpha} x} \tag{2}$$

and breathers which may be considered as soliton-antisoliton bound states. On the semiclassical level there is a sequence of breathers b_k with masses [3]

$$m_k = 2m \sin \frac{k\pi}{2\lambda}\,, \quad k = 1, 2, \ldots < \lambda \tag{3}$$

where m is the soliton mass and λ is a parameter related to the coupling constant:

$$\lambda = \frac{8\pi}{\beta^2} - 1. \tag{4}$$

The soliton is a coherent state

$$|\, sol\,\rangle = \exp \int dx\, \phi(x)\; i\overleftrightarrow{\partial_t}\; \phi_{sol}(x)\, |\, 0\,\rangle \tag{5}$$

or

$$\langle\, sol\, |\, \phi(x)\, |\, sol\,\rangle = \phi_{sol}(x)$$

which means that the soliton is an eigenstate of the annihilation operator

$$\phi^{(+)}(x)\, |\, sol\,\rangle = |\, sol\,\rangle\, \phi^{(+)}_{sol}(x)\,. \tag{6}$$

On the quantum level the fundamental boson corresponding to the field $\phi(x)$ is the lowest breather b_1. Equation (6) means that the soliton is a soliton-breather bound state.

ii) The massive Thirring model (MT) model defined by the Lagrangian

$$L = \bar{\psi}(i\gamma\partial - m)\psi - \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)^2 \tag{7}$$

describes the interaction of fermions f and $\bar{f}$. For $g > 0$ there exist fermion-antifermion bound states b_k.

The famous equivalence of the quantum Sine-Gordon and the massive Thirring model due to Coleman [4] says: identify the SG-solitons and breathers with the MT-fermions and bound states, respectively, relate the SG-field to the MT-current by

$$\beta\epsilon^{\mu\nu}\partial_\nu\phi = -2\pi\bar{\psi}\gamma^\mu\psi$$

and the coupling constants by

$$\lambda = \frac{8\pi}{\beta^2} - 1 = 1 + \frac{2g}{\pi}\,. \tag{8}$$

iii) The Gross-Neveu (GN) model [5] defined by the Lagrangian

$$L = \sum_{I=1}^{N} \bar{\psi}_i i\gamma\partial\psi - \frac{g^2}{2}\left(\sum_{i=1}^{N} \bar{\psi}_i\psi_i\right)^2$$

describes an $O(N)$-symmetric interaction of N (even) selfconjugate fermions f_i with a dynamically generated mass m_1. There are $2^{N/2}$ states k_α which are $O(N)$-isospinors [6]. The kink-kink bound states $(i_1 \dots i_j)_k$ ($j = 0, 2, \dots, k$, for even k, $j = 1, 3, \dots, k$, for odd k) which are antisymmetric tensors, have the spectrum eq. (3) with $\lambda = \frac{N}{2} - 1$ [7]. The lowest bound states ($k = 1$) are the fundamental fermions f_i, the higher ones may also (as for the SG model) be considered as bound states of k fundamental fermions.

iv) There are further models to which the method used in this paper can be applied, e.g. the $O(N)$-nonlinear σ-model [8], a chiral $SU(N)$ model [9], a $Z(N)$-Ising model in the scaling limit $T \to T_c$ [10].

1.2 Infinitely many conservation laws

The models presented in the introduction are in some sense simple because they have soliton behaviour, this means they possess infinitely many conservation laws.

1.2.1 The classical case [11]

Let us take the SG-model as an example. There exist an infinite set of local local conservations laws

$$\partial_\mu J_k^\mu(x) = 0\ , \quad k = 1, 3, 5, \dots \tag{9}$$

This can be seen as follows: Define recursively as a formal power series

$$\sigma(x,\epsilon) = \sum_{k=1}^{\infty} \epsilon^k \sigma_k(x) = \left(1 - \epsilon\,\partial_+ - \frac{\epsilon}{2}\,\sigma\partial_+\,\phi\right)^{-1}\frac{\epsilon}{2}\,\partial\phi$$

and

$$(J_+, J_-) = \sum_{k=1}^{\infty} \epsilon^k\,(J_{k+}J_{k-}) = (\sigma\,\partial_+\phi, \epsilon(\sigma\sin\phi - \cos\phi))$$

where

$$\partial_\pm = \partial_0 \pm \partial_1\ , \quad J_\pm = J_0 \pm J_1.$$

Then for any solution of the SG-equation $\Box\phi + \sin\phi = 0$ eq. (9) is fulfilled and the quantities

$$Q_k = \int dx\, J_k^0 = \int dx\,\left(i\partial_0\phi\,\partial_k\phi + O(\phi^3)\right) \tag{10}$$

are conserved, $\dot{Q}_k = 0$. The contribution $O(\phi^3)$ vanishes on asymptotic states. If we consider the classical theory as the tree approximation of the quantum theory we obtain

$$Q_k \,|\, p_1, \ldots, p_n \,\rangle^{in} = \sum_{i=1}^{n} (p_+)^k \,|\, p_1, \ldots, p_n \,\rangle^{in} \tag{11}$$

for $n = 1, 3, 5, \ldots$ (for n=even Q_n vanishes).

Since the S-matrix commutes with Q and eq. (11) holds for any k for a scattering process the set of incoming and outgoing momenta are equal

$$\{p_1, \ldots, p_n\}^{in} = \{p'_1, \ldots, p'_n\}^{out} \tag{12}$$

This means absence of particle production and only momenta exchange. For some models there exist also non local conservation laws [12] with similar consequences.

1.2.2 Quantization

The BPHZ-quantization [13] of the conservation laws (9) apparently produce anomalies which cancel after redefinition of the currents J_k^μ [14]. The nonlocal charges are also conserved in the quantized models [15]. In summary we can say the infinitely many conservation laws survive quantization.

1.2.3 Consequences for scattering

Since eqs. (11) and (12) are also true for the quantum models we have absence of particle production and only momentum exchange between particles with the same mass. Furthermore one can show [16] that the n-particle S-matrix factorizes into a product of two-particle S-matrices

$$S^{(n)}(p_1, \ldots, p_n) = \sum_{i<j} S^{(2)}(p_i, p_j) \tag{13}$$

For non-vanishing reflection the factors on the r.h.s. do not commute and we have to specify the order. For $n = 3$ and $p_1^1 > p_2^1 > p_3^1$ there are two possible orders (with $S^{(2)}(p_i, p_j) = S_{ij}$)

$$S^{(3)} = S_{12}\, S_{13}\, S_{23} = S_{23}\, S_{13}\, S_{12} \tag{14}$$

This so called factorization equation give relations between the two-particle scattering amplitudes which allow to calculate the S-matrix exactly [1].

1.3 Construction of the S-matrix

1.3.1 General procedure

We calculate the S-matrix from the assumptions:

i) Factorization

ii) Qualitative knowledge of the one-particle spectrum; i.e. we assume e.g. for the MT-model the existence of a coupling region with no bound states, for the SG-model the existence of a (b_1-b_1) bound state and the soliton to be a (soliton-b_1 bound state.

iii) "Minimality", i.e. absence of redundant poles and zeros in the physical sheet of the transmission amplitudes (this is true for potential scattering [17]).

iv) Unitarity, crossing, internal symmetry $(O(N), U(N), Z(N))$, e.g. for the MT-model: $f, \bar{f}$ = fundamental representation of $U(1)$; for the SG-model: b_1 = selfconjugate isosinglett.

v) Asymptotically good behaviour for $p_1 p_2 \to \infty$

$$S^{(2)} = o\left(\exp \frac{p_1 p_2}{m^2}\right).$$

By these assumptions the S-matrix is "uniquely" determined. "Uniquely" means we have

for $U(1) \equiv O(2)$	a one-parametric set of solutions [1],
for $O(N)$, $(n > 2)$	two solutions [1], [18],
for $U(N)$, $(N > 1)$	five solutions [18],
for $Z(N)$	one solution [10].

A sketch of the proof [1]:

1. The factorization equation (14) implies e.g. for the MT-model fermion-antifermion amplitudes
$$h(\alpha+\beta) = h(i\pi+\alpha)\, h(\beta) + h\alpha)\, h(i\pi-\beta) \tag{15}$$
where $h(\theta)$ is the ration of transmission and reflection amplitudes and θ is the rapidity difference defined by $p_1 p_2 = m^2 \cosh\theta$. The solution of eq. (15) is
$$h(\theta) = \frac{\sinh \lambda\theta}{\sinh \lambda i\pi} \tag{16}$$
where λ is a free parameter.

2. Unitarity, crossing, "minimality" and eq. (16) imply uniquely the two particle S-matrix.

1.3.2 Bound state scattering [19,20]

Let b be a bound state with mass m_b of two particles α and β, then the two particle S-matrix has a pole at $\theta = ia$ where a is given by

$$m_b^2 = m_\alpha^2 + m_\beta^2 + 2m_\alpha m_\beta \cos a\,. \tag{17}$$

The S-matrix for the scattering of the bound state b with a particle γ is obtained by considering the three-particle S-matrix for the scattering of $\alpha(p_1), \beta(p_2), \gamma(p_3)$ at the pole $\theta_{12} = ia$. Factorization and unitarity imply [20]

$$S_{1+2,3} = |R_{12}|^{-1/2} S_{23} S_{13} |R_{12}|^{1/2} \tag{18}$$

where

$$R_{12} = \operatorname*{Res}_{\theta=ia} S_{12}\,.$$

1.3.3 Examples

i) Massive Thirring model

All states of two particles with momenta p_1 and p_2 are $ff, f\bar{f}, \bar{f}f, \bar{f}\bar{f}, fb_1, \ldots, b_1b_1, \ldots$. Since reflection can only appear for two different particles with the same mass, the two-particle S-matrix has the form

$$S_{12} = \begin{pmatrix} u &&&&&&& \\ & t & r &&&&& \\ & r & t &&&&& \\ &&& u &&&& \\ &&&& S_{fb_1} &&& \\ &&&&& \ddots && \\ &&&&&& S_{b_1b_1} & \\ &&&&&&& \ddots \end{pmatrix}. \tag{19}$$

For $g < 0$ there are no bound states. The $U(1)$-S-matrix obtained from factorization, unitarity, crossing and minimality of section 3.1 has no pole in the physical sheet $0 < \operatorname{Im}\theta < \pi$ for $0 < \lambda < 1$. Thus we propose first in this coupling region

$$(u,t,r) = \left(1, \frac{\sinh\lambda\theta}{\sinh\lambda(i\pi-\theta)}, \frac{\sinh\lambda i\pi}{\sinh\lambda(i\pi-\theta)}\right) \exp\int_0^\infty \frac{dx}{x} \frac{\sinh\frac{x}{2}(1-1/\lambda)}{\sinh\frac{x}{2\lambda}\cosh\frac{x}{2}} \sinh x\frac{\theta}{i\pi}. \tag{20}$$

For $\lambda > 1$ the $f\bar{f}$-transmission amplitude $t(\theta)$ has poles at

$$\theta = i\pi(1-k\lambda)\ , \quad k = 1, 2, \ldots < \lambda. \tag{21}$$

These poles correspond to $f\bar{f}$-bound states b_k with masses

$$m_k = 2m \sin \frac{k\pi}{2\lambda}. \tag{22}$$

This agrees with the WKB-spectrum eq. (3) for the SG-model if we identify

$$\lambda = \frac{8\pi}{\beta^2} - 1 = 1 + \frac{2g}{\pi}\,.$$

The amplitudes given by eq. (20) have been first proposed by Zamolodchikov [21] by different arguments. They have been checked in perturbation theory at $g \to 0$ and $\beta \to 0$ [21,22].

The scattering of fermions and bound states b_k can be calculated by means of eq. (18) [19]. E.g. for b_1 we obtain

$$S_{fb_1}(\theta, \lambda) = \frac{\sinh\theta + i \sin\frac{\pi}{2}(1 + 1/\lambda)}{\sinh\theta - i \sin\frac{\pi}{2}(1 + 1/\lambda)}\,. \tag{23}$$

The pole at $\theta = \frac{1}{2} i\pi(1+1/\lambda)$ correspond to the bound state $f = (fb_1)$ which is anticipated by eq. (6).

The scattering of two bound states can be derived from eqs. (18) and (23). The result is e.g. for b_1-b_1:

$$S_{b_1 b_1}(\theta, \lambda) = \frac{\sinh\theta + i \sin\frac{\pi}{\lambda}}{\sinh\theta - i \sin\frac{\pi}{\lambda}}\,. \tag{24}$$

The pole at $\theta = i\pi/\lambda$ correspond to the bound state $b_2 = (b_1 b_1)$. (The generalization of this formula is $b_{i+j+k+\ldots} = (b_i + b_j + b_k + \ldots)$.) The amplitudes can be checked in SG-perturbation theory at $\beta \to 0$.

ii) Sine-Gordon model

In the last section we have derived the complete S-matrix of the MT-model alias the SG-model by starting with the soliton S-matrix and then calculating the breather scattering. Now we will do it the other way round without using directly the factorization equation (15) but the property (6) that the soliton is a bound state of itself and a breather.

For the breather-breather S-matrix the absence of particle production, unitarity, crossing, "minimality" and the existence of a $(b_1 b_1)$-bound sate imply for the scattering of the selfconjugate iso-singlett b_1 the S-matrix (24). The existence of a two particle bound state $b_2 = (b_1 b_1)$ with mass $m_2 = m_1 \sin(\pi/\lambda)$ and factorization imply the existence of a k-particle bound state

$$b_k = (b_1 \ldots b_1) = \ldots = (b_{i_1} \ldots b_{i_j}) = (b_{k-1} b_1) \quad \text{with} \quad i_1 + \ldots + i_j = k$$

and the spectrum (3) [23]. This can be seen inductively as follows: The S-matrix of b_{k-1} and b_1 is given by eq. (18) to be

$$S_{b_{k-1}b_1}(p_1+\ldots+p_{k-1},p_k) = S_{b_1b_1}(p_1,p_k)\ldots S_{b_1b_1}(p_{k-1},p_k) \tag{25}$$

where (for suitable ordered momenta) the rapidity differences are

$$\theta_{jj+1} = i\pi/\lambda \quad (j=1,\ldots,k-2)\,.$$

The bound state pole of $S_{b_{k-1}b_1}$ is again at $\theta_{k-1,k} = i\pi/\lambda$. Therefore the mass of b_k is

$$m_k = m_1 \left| \sum_{j=1}^{k-1} e^{i\pi j/\lambda} \right| = m_1 \frac{\sin\frac{k\pi}{2\lambda}}{\sin\frac{\pi}{2\lambda}} \tag{26}$$

which agrees with eq. (3). Note that there are two cases:

a) there exist kinks with mass

$$m = m_1/\left(2\sin\pi/(2\lambda)\right)$$

then $k=1,2,\ldots<\lambda$ as for the massive Thirring model and the Gross-Neveu model or

b) there exist no kinks then $k=1,2,\ldots<2\lambda$ as for the chiral $SU(N)$ model [9] and a $Z(N)$-Ising model [10].

For the scattering of a soliton and a breather the property that the soliton is a bound state of itself and a breather together with eq. (18) and unitarity implies

$$\begin{aligned} S_{fb_1}(\theta) &= S_{fb_1}(\theta+i\pi/\lambda)\,S_{b_1b_1}(\theta-i\tfrac{\pi}{2}(1-1/\lambda)) \\ &= S_{b_1b_1}(\theta+i\tfrac{\pi}{2}(1-1/\lambda))\,S_{fb_1}(\theta-i\pi/\lambda) \end{aligned} \tag{27}$$

These equations have the unique "minimal" solution (23). For the soliton-soliton scattering the same arguments lead to

$$\begin{aligned} u(\theta) &= u(\theta+i\pi/\lambda)\,S_{fb_1}(\theta-i\tfrac{\pi}{2}(1-1/\lambda)) \\ &= S_{fb_1}(\theta+i\tfrac{\pi}{2}(1-1/\lambda))\,u(\theta-i\pi/\lambda) \end{aligned} \tag{28}$$

The "minimal" solution (with $u(0)=1$) is obtained as follows:

$$\begin{aligned} u(\theta) &= \exp\frac{1}{2\pi i}\int_C \frac{dz}{\sinh(z-\lambda\theta)}\,\frac{\sinh\lambda\theta}{\sinh z}\,\ln u(z/\lambda) \\ &= \exp\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dz}{\cosh(z-\lambda\theta)}\,\frac{\sinh\lambda\theta}{\cosh z}\,\ln\frac{u(z/\lambda-i\frac{\pi}{2\lambda})}{u(z/\lambda+i\frac{\pi}{2\lambda})} \end{aligned} \tag{29}$$

where $\mathcal{C}$ encloses the strip $-\pi/2 \le \mathrm{Im}\, z \le \pi/2$.

Using eq.(28) we can express $u(\cdot)$ in terms of $S_{fb_1}(\cdot)$. The result agrees with eq. (20) which can be understood as another proof of Coleman's equivalence.

iii) Gross-Neveu model

The same approach as in ii) can be used to calculate the fermion-kink and kink-kink scattering for the GN-model. The direct method using factorization is complicated because of group theoretical problems [6]. From factorization, unitarity, crossing, "minimality" and the existence of an (ff)-bound state in the $O(N)$-isoscalar and antisymmetric tensor channel one obtains for the scattering of fundamental fermions [24]

$$f_i(p_1) + f_j(p_2) \longrightarrow f_k(p_1) + f_l(p_2)$$

$$\begin{aligned} {}_{kl}S_{ij}(\theta) &= \Big(\delta_{ik}\delta_{jl} - \tfrac{N-2}{2\pi i}(i\pi-\theta)\,\delta_{il}\delta_{jk} - \tfrac{N-2}{2\pi i}\theta\,\delta_{ij}\delta_{kl}\Big) \\ &\times S_{b_1b_1}(\theta,\lambda=N/2-1)\,\exp\int_0^\infty \frac{dx}{x}\,\frac{e^{-2x/(N-2)}-1}{\cosh\frac{x}{2}}\cosh x(\tfrac{\theta}{i\pi}-\tfrac{1}{2}) \end{aligned} \tag{30}$$

where $S_{b_1b_1}(\theta,\lambda)$ is given by eq. (24). Analogously to the SG-case we obtain for the fermion-kink scattering[1]

$$f_i(p_1) + k_\alpha(p_2) \longrightarrow f_j(p_1) + k_\beta(p_2)$$

$$\begin{aligned} {}_{j\beta}S_{i\alpha}(\theta) &= -\Big(\delta_{ij}\delta_{\alpha\beta} - \tfrac{1}{N}\,\gamma^j_\beta\gamma^i_\alpha - \tfrac{2(N-2)\theta-i\pi N}{2(N-2)\theta+i\pi N}\,\tfrac{1}{N}\,\gamma^i_\beta\gamma^j_\alpha\Big) \\ &\times S_{fb_1}(\theta,\lambda=N/2-1)\,\exp\int_0^\infty \frac{dx}{x}\,\frac{e^{-x(1+1/(N-2))}}{\cosh\frac{x}{2}}\sinh x\tfrac{\theta}{i\pi} \end{aligned} \tag{31}$$

where $S_{fb_1}(\theta,\lambda)$ is given by eq. (23) and the γ^i_α are the usual γ matrices of $O(N)$, satisfying

$$\gamma^i\gamma^j + \gamma^j\gamma^i = \delta_{ij}.$$

Again analogously to the SG-case we obtain for kink-kink scattering[1]

$$k_\alpha(p_1) + k_\beta(p_2) \longrightarrow k_\gamma(p_1) + k_\delta(p_2)$$

$${}_{\gamma\delta}S_{\alpha\beta}(\theta)\sum_{n=0}^{N}\frac{1}{n!2^{N/2}}\,\sigma^{(n)}_{\delta\alpha}\otimes\sigma^{(n)}_{\gamma\beta}\,u_n(\theta) \tag{32}$$

where $u_n(\theta) = (-1)^{N/2}\,u_{N-n}(\theta)$ and

$$\begin{aligned} u_n(\theta) &= u(\theta,\lambda=N/2-1)\,\exp\int_0^\infty \frac{dx}{x}\,\frac{e^{-x(1+1/(N-2))}}{\cosh\frac{x}{2}\sinh x\frac{2}{N-2}} \\ &\times\Big(\sinh\tfrac{x}{2}(1+\tfrac{2}{N-2}) - 2\,e^{x(n-1/(N-2))}\cosh\tfrac{x}{2}\sinh x\tfrac{n}{N-2}\Big)\sinh x(\tfrac{\theta}{i\pi}-1) \end{aligned} \tag{32a}$$

[1] More details will be published elsewhere [24].

for n even $\leq N/2$,

$$u_n(\theta) = u(\theta,\lambda = N/2-1)\,\exp\int_0^\infty \frac{dx}{x}\,\frac{e^{-x(1+1/(N-2))}}{\cosh\frac{x}{2}\sinh x\frac{2}{N-2}} \\ \times\Big(\sinh\tfrac{x}{2}(1-\tfrac{2}{N-2}) - 2\,e^{x(n/(N-2))}\cosh\tfrac{x}{2}\sinh x\tfrac{n-1}{N-2}\Big)\sinh x(\tfrac{\theta}{i\pi}-1) \tag{32b}$$

for n odd $\leq N/2$ and

$$\sigma^{(n)}_{\delta\alpha}\otimes\sigma^{(n)}_{\gamma\beta} = \frac{1}{n!}\sum_{\pi\in S_n}(-1)^\pi\left(\gamma^{\pi(i_1)}\dots\gamma^{\pi(i_n)}\right)_{\delta\alpha}\otimes\frac{1}{n!}\sum_{\pi'\in S_n}(-1)^{\pi'}\left(\gamma^{\pi'(i_1)}\dots\gamma^{\pi'(i_n)}\right)_{\gamma\beta}.$$

The SG-amplitude $u(\theta,\lambda)$ is given by eq. (20).

2 The fields

We will calculate "generalized form factors"

$$^{out}\langle\, p_1',\dots\,|\,O(x)\,|\,p_1,\dots\,\rangle^{in}$$

for local operators and Green's functions

$$\langle\, T\phi(x_1)\dots\phi(x_n)\,\rangle\,.$$

These problems are up to now only solved for special simple cases.

2.1 Introduction

The form factor problem is simple for the $Z(N)$-Ising model mentioned in chap. 1.1. It is completely solved for $N=2$. We will give a short introduction to these models.

$Z(N)$-symmetry means that the order variable fulfills

$$\sigma^N = 1 \quad\text{or}\quad \sigma^\dagger = \sigma N - 1\,.$$

If b_1 is the fundamental particle to the field $\sigma(x)$ and there exist a two particle bound state $b_2=(b_1b_1)$ then $\sigma^\dagger=\sigma^{N-1}$ implies that the anti-particle is an $N-1$ bound state

$$\bar b_1 = (b_1\dots b_1)\,.$$

This fact together with factorization, unitarity, crossing and "minimality" leads to the S-matrix [10]

$$S_{b_1b_1}(\theta) = \frac{\sinh\frac12(\theta+2\pi i/2)}{\sinh\frac12(\theta-2\pi i/2)} \tag{33}$$

and the mass spectrum for the particles $b_k = \bar{b}_{N-k}$ is

$$m_k = m_1 \frac{\sinh(k\pi/N)}{\sinh(\pi/N)}\,. \tag{34}$$

For the $Z(2)$-Ising model we get a self conjugate isosinglett $b_1 = \bar{b}_1$ and

$$S_{bb} = -1\,. \tag{35}$$

2.2 Watson's Theorem

We derive a set of equations for matrix elements of local operators which follow from general principles of quantum field theory, "maximal" analyticity and the S-matrix factorization. For simplicity we first consider the case where we have only one kind of boson in the model and a hermitian operator $O(x)$. If we define (with $p_1 p_2 = m^2 \cosh\theta$)

$$\langle\, 0 \,|\, O(0) \,|\, p_1, p_2 \,\rangle^{in} = F(\theta) \tag{36}$$

it follows from

- CTP-invariance: $\langle\, 0 \,|\, O(0) \,|\, p_1, p_2 \,\rangle^{out} = F(-\theta)$
- unitarity and factorization: $F(\theta) = \langle\, 0 \,|\, O(0) \,|\, p_1, p_2 \,\rangle^{out}\, S(\theta)$
- and crossing: $\langle\, p_1 \,|\, O(0) \,|\, p_2 \,\rangle = F(i\pi - \theta)$.

Hence we have Watson's equations

$$\begin{aligned} F(\theta) &= F(-\theta)\, S(\theta) \\ F(i\pi - \theta) &= F(i\pi + \theta)\,. \end{aligned} \tag{37}$$

The corresponding equations for the general case where α_i denote different kinds of particles read

$$\begin{aligned} F_{\alpha_1\dots\alpha_n}(\theta_{ij}, i\pi - \theta_{rs}, \theta_{kl}) &= {}^{out}\langle\, \alpha_1(p_1), \dots, \alpha_m(p_m) \,|\, O(0) \,|\, \dots, \alpha_n(p_n) \,\rangle^{in} \\ &= {}_{\alpha_1\dots\alpha_m}S^{(m)}_{\alpha'_1\dots\alpha'_m}(\theta_{ij}) F_{\alpha'_1\dots\alpha'_n}(-\theta_{ij}, i\pi + \theta_{rs}, -\theta_{kl})\, {}_{\alpha'_{m+1}\dots\alpha'_n}S^{(n-m)}_{\alpha_{m+1}\dots\alpha_n}(\theta_{kl}) \end{aligned} \tag{38}$$

where $1 \le i < j \le m$, $1 \le r \le m < s \le n$, $m < k < l \le n$ and $0 \le m \le n$. Solutions of these equations are only known for simple cases.

2.3 Construction of form factors

The problem of constructing form factors has been solved for models with arbitrary number of kinds of particles for the case $n = 2$. This means usual form factors, i.e. one particle expectation values of local operators, are known exactly for e.g. the massive Thirring [25], the nonlinear σ-, and the Gross-Neveu model [26]. Furthermore arbitrary n-particle matrix elements are known for the $Z(N)$-Ising model in the scaling limit with one kind of particles and $S^{(2)} = -1$ [27]. There are two distinct problems:

2.3.1 The "matrix problem"

The set of matrix equations (38) has not been solved for the general case. For $n = 2$ we diagonalize $S^{(2)}$ and get for each eigenvalue the simple equations (37).

Theorem: [26]

A function $F(\theta)$ fulfilling eqs. (37) is uniquely (up to a normalization) determined by the position of poles (and zeros) at $\theta = ia_k$ in the physical strip $0 \leq \text{Im } \theta \leq \pi$:

$$F(\theta) = K(\theta)\, F^{min}(\theta) \tag{39}$$

where

$$K(\theta) = const \left(\prod_k \sinh \frac{1}{2}(\theta - ia_k)\, \sinh \frac{1}{2}(\theta + ia_k) \right)^{-1} \tag{39a}$$

and

$$F^{min}(\theta) = \exp \frac{1}{4\pi i} \int_{-\infty}^{\infty} \frac{dz}{\sinh \frac{1}{2}(z-\theta)} \frac{\cosh \frac{1}{2}\theta}{\cosh \frac{1}{2} z} \ln S(z) \tag{39b}$$

For vanishing reflection $S^{(n)}$ is diagonal and a solution of the general equation (38) is:

$$F_{\alpha_1 \ldots \alpha_n}(\theta_{12}, \ldots) = K_{\alpha_1 \ldots \alpha_n}(\theta_{12}, \ldots) \prod_{i<j} F^{min}_{\alpha_i \alpha_j}(\theta_{ij}) \tag{40}$$

where K is a solution of eqs. (38) with $S = 1$.

2.3.2 The "pole problem"

The poles in the physical region of the function

$$F_{\alpha_1 \ldots \alpha_n}(\theta_{12}, \ldots) = \langle\, 0 \,|\, O(0) \,|\, \alpha_1(p_1), \ldots, \alpha_n(p_n)\, \rangle^{in}$$

which are contained in the function K are determined by one-particle states in all subchannels $\alpha_{i_1}, \ldots, \alpha_{i_j}$. If we make the minimality assumption that there are no further so called redundant poles and no zeros in the physical region of F, we can make proposals for exact form factors, which can then be checked by perturbation theory.

2.4 Examples

2.4.1 Sine-Gordon model

The electro magnetic SG-soliton form factor is defined by

$$\langle\, f(p_1)\,|\, j^\mu(0)\,|\, f(p_2)\,\rangle = \bar{u}(p_1)\gamma^\mu u(p_2)\, F(i\pi - \theta)$$

where $p_1 p_2 = m^2 \cosh\theta$. Since there are no bound states for $g < 0$, we propose in this region F to be the minimal solution of Watson's equations (37) with the negative C-parity S-matrix eigenvalue $t - r$ given by eq. (20). For $g > 0$ we take the analytic continuation [25]:

$$F(i\pi - \theta) = \frac{\cosh \frac{1}{2}\theta}{\cosh \frac{1}{2}\lambda\theta} \exp \int_0^\infty \frac{dx}{x} \frac{\sinh \frac{x}{2}(1 - 1/\lambda)}{\sinh \frac{x}{2\lambda} \cosh \frac{x}{2}} \frac{\sin^2 \frac{x\theta}{2\pi}}{\sinh x}\,. \tag{41}$$

This formula was checked [25] in perturbation theory $g \to 0$ and in the classical limit $\beta \to 0$ for which one gets the classical soliton solution eq. (2).

2.4.2 $Z(2)$-Ising model

For the $Z(2)$-Ising model in the scaling limit the problem of calculating arbitrary matrix elements of the order variable

$$\langle\, 0\,|\, \sigma(0)\,|\, p_1, \ldots, p_n\,\rangle^{in} = F^{(n)}(\theta_{12}, \ldots, \theta_{n-1,n}) \tag{42}$$

has been solved completely [27]. The minimal solution of Watson's equations with $S^{(2)} = -1$ is

$$F^{min}(\theta) = -i \sinh \frac{\theta}{2}\,. \tag{43}$$

Since there is no reflecting we can use formula (40)

$$F^{(n)}(\theta_{12}, \ldots) = K^{(n)}(\theta_{12}, \ldots) \prod_{i<j} F^{min}(\theta_{ij})\,. \tag{44}$$

The function K contains the poles from one-particle states in all subchannels $p_{i_1} + \ldots + p_{i_j}$. For $n = 3$ we obtain

$$K^{(3)} = \frac{const}{(p_1 + p_2 + p_3)^2 - m^2} = \frac{Z^{(3)}}{\cosh \frac{1}{2}\theta_{12}\, \cosh \frac{1}{2}\theta_{13}\, \cosh \frac{1}{2}\theta_{23}}\,. \tag{45}$$

The constant $Z^{(3)}$ is determined to be $Z^{(3)} = 2$ by the requirement that $F^{(3)}$ reproduces the correct S-matrix $S^{(2)} = -1$. For $n \geq 5$ the arguments are just slightly more involved. Again the function $K^{(n)}$ must have poles at $(p_i + p_j + p_k)^2 = m^2$ where any set of particles i, j, k are combined to give a one-particle intermediate state. However, the residues of "higher poles" where more than three particles build up a one-particle state must vanish, since the presence of such poles would be in contradiction to the absence of particle production. Hence the minimality hypothesis implies that $K^{(n)}$ has the form

$$K^{(n)} = \frac{const\ \left(\prod_{i<j}(p_i+p_j)^2\right)^{\frac{n-3}{2}}}{\prod_{i<j<k}\left((p_i+p_j+p_k)^2-m^2\right)} = \frac{Z^{(n)}}{\prod_{i<j}\cosh\frac{1}{2}\theta_{ij}} \tag{46}$$

The numerator in eq. (46) is necessary to modify the severe singularities of the denominator at $p_i + p_j = 0$ in the matrix element (42) into simple poles. The constant $Z^{(n)}$ can again be determined to be $Z^{(n)} = 2^{(n-1)/2}$ by crossing and taking the off shell leg $\sigma(x)$ in eq. (42) on shell. Finally we obtain

$$\langle\, 0 \,|\, \sigma(0) \,|\, p_1, \ldots, p_n \,\rangle^{in} = (2i)^{\frac{n-1}{2}} \prod_{i<j} \tanh\frac{1}{2}\theta_{ij}\,. \tag{47}$$

By crossing only the connected part of the matrix element is given by analytic continuation. The complete matrix element can be determined as follows, e.g. for $n = 3$:

For the transition

$$\langle\, 0 \,|\, \sigma(0) \,|\, p_1, p_2, p_3 \,\rangle^{in} \to \langle\, p_1 \,|\, \sigma(0) \,|\, p_2, p_3 \,\rangle^{in}$$

we have to replace

$$\theta_{1i} \to i\pi - \theta_{1i}\,.$$

This means

$$\tanh\frac{1}{2}\theta_{1i} \to \mathcal{P}\left(\tanh\frac{1}{2}\theta_{1i}\right)^{-1} + f\,\delta(p_1 - p_i)$$

where $\mathcal{P}$ is the principal part and the δ-function part is determined by the disconnected part of the matrix element. It can be calculated by considering the asymptotic LSZ-limit $t \to \pm\infty$:

$$\begin{array}{lrcl} t \to +\infty & \langle\, p_1 \,|\, a^{out}(p) \,|\, p_2, p_3 \,\rangle^{in} & = & \left(\delta_{p_1 p_2}\,\delta_{p p_3} + 2 \leftrightarrow 3\right) S^{(2)} \\ t \to -\infty & \langle\, p_1 \,|\, a^{in}(p) \,|\, p_2, p_3 \,\rangle^{in} & = & \delta_{p_1 p_2}\,\delta_{p p_3} + 2 \leftrightarrow 3 \end{array}\,.$$

We find that f is proportional to $1 + S^{(2)}$. Since for the $Z(2)$-Ising model $S^{(2)} = -1$ there is only the principal part term. Hence

$$\begin{aligned} &{}^{out}\langle\, p_1, \ldots, p_m \,|\, \sigma(0) \,|\, \ldots, p_n \,\rangle^{in} \\ &= (2i)^{\frac{n-1}{2}} \prod_{i<j\le m} \tanh\frac{1}{2}\theta_{ij} \prod_{r\le m<s} \mathcal{P}\Big(\tanh\frac{1}{2}\theta_{rs}\Big)^{-1} \prod_{m<k<l} \tanh\frac{1}{2}\theta_{kl} \end{aligned} \tag{48}$$

is the complete matrix element of the order variable.

2.5 Green's functions

Till now the problem of constructing Green's functions has only been solved for the simplest field theory with soliton behaviour, the $Z(2)$-Ising model in the scaling limit $T \to T_c$ [27]. Having deduced all matrix elements of the σ-field, we can immediately write down expressions for Green's functions. For example the two point function is

$$\tau(p) = \int d^2x\, e^{ipx} \langle\, T\sigma(x)\,\sigma(0)\,\rangle = i\int d\kappa^2\, \frac{\rho(\kappa^2)}{p^2 - \kappa^2 + i\epsilon}$$

with the spectral function ρ given by

$$\rho(p^2) = \sum_{n\,\mathrm{odd}} \frac{1}{n!} \int \frac{d\theta_1}{4\pi} \cdots \int \frac{d\theta_n}{4\pi}\, 2\pi\, \delta^{(2)}\left(p - \sum p_i\right) \prod_{1 \le i < j \le n} \tanh^2 \frac{1}{2}\theta_{ij}\,.$$

There are similar formulae for the n-point Green's functions which are all in agreement with those derived in ref. [28]. This agreement supports the validity of the minimality hypothesis and gives some confidence in the feasibility of the bootstrap program. The determination of correlation functions for one of the more involved soliton field theories remains, however, a challenging open problem.

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