# Construction of Green's functions from an exact $S$ matrix

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The bootstrap program for determining Green's functions from an exact $S$ matrix is carried out for the simplest soliton field theory of a scalar field with $S$-matrix operator $S=(-1)^{N(N-1)/2}$, where $N$ is the total number operator. Despite the formal simplicity of the $S$ matrix, the Green's functions derived have a rich structure. The results can be checked since this field theory is none other than that of the order variable of the Ising model in the scaling limit above the critical temperature.

For two-dimensional field theories with soliton behavior the dynamics is governed by an infinite number of higher conservation laws. In particular, for scattering processes they imply absence of particle production and factorization of the $n$-particle $S$ matrix. For the simplest case of a single species of particle of given mass (=1) this means (assuming parity conservation)

$$^{\text{out}}\langle p_1'\cdots p_m'|p_1\cdots p_n\rangle^{\text{in}}=\delta_{mn}S^{(n)\,\text{in}}\langle p_1'\cdots p_n'|p_1\cdots p_n\rangle^{\text{in}}$$

with (1)

$$S^{(n)}=\prod_{1\le i<j\le n}S^{(2)}(|\theta_i-\theta_j|),$$

where $\theta_i$ are the rapidities, $p_i=(\cosh\theta_i,\sinh\theta_i)$, and $S^{(2)}(\theta)$ is the two-particle $S$-matrix element satisfying elastic unitarity for all energies. It is the boundary value of a real analytic function $S(\theta)$. The working hypothesis is that $S(\theta)$ is meromorphic in the $\theta$ plane. The properties real analyticity, crossing symmetry, and unitarity can then be written (by analytic continuation for all $\theta$)

$$\begin{aligned}&S^*(\theta)=S(-\theta^*),\\&S(\theta)=S(i\pi-\theta),\\&S(\theta)\,S(-\theta)=1.\end{aligned}\qquad(2)$$

There always exists a minimal or a finite class of minimal nontrivial $S$ matrices consistent with the above structure. Nontrivial means $S\neq 1$ and by minimal is meant that the $S$ matrix having the minimal number of zeros and poles in the physical strip $(0<\operatorname{Im}\theta<\pi)$ apart from the given (two-particle) bound-state poles. (In all cases considered so far resonances have also been assumed to be absent.) For the case described above the minimal solution of Eqs. (2) is

$$S(\theta)=\pm\prod_a\frac{\sinh\theta+\sinh i\alpha_a}{\sinh\theta-\sinh i\alpha_a},\qquad(3)$$

where the poles at $\theta=i\alpha_a$ all represent two-particle bound states. In more complicated cases when the particles belong to a nontrivial representation of an internal-symmetry group, there are in general many invariant amplitudes. The constraints arising from a consistent formulation of factorization in these cases are, however, so stringent that they interrelate the invariant amplitudes so that the minimal $S$ matrices can again be completely determined. This has been done for a variety of models and the minimal $S$ matrices have so far stood all tests.[1]

It has often been postulated that such soliton field theories are in fact soluble. The program that has been advocated is just the old bootstrap program. This involves explicitly solving the Watson equations for the field matrix elements and subsequently obtaining representations for the Green's functions by summing over a complete set of intermediate asymptotic states. The program is, however, complicated and comparatively little progress has been made for the interesting soliton field theories. In particular, apart from the matrix problems which arise in solving the Watson equations, minimality assumptions are made, the rationale for which is not completely understood.

It is the purpose of this paper to completely carry out the bootstrap program for the simplest soliton field theory. This is the $Z_2$-invariant field theory of a single (self-conjugate) boson field $\sigma(x)$ whose associated quanta have the minimal nontrivial factorizing $S$ matrix with no bound states. As is seen from Eq. (3) $S(\theta)$ is then simply given by

$$S=-1,\qquad(4)$$

and from Eq. (1) it follows that the $n$-particle $S$-matrix element is

$$S^{(n)}=\prod_{1\le i<j\le n}(-1)=(-1)^{n(n-1)/2}.\qquad(5)$$

This field theory is in fact none other than that of the order variable of the Ising model in the scaling limit[2] above the critical temperature, as has been established by Sato, Miwa, and Jimbo.[4,5] It is extremely satisfying that starting from the nearly trivial on-shell data (5) the bootstrap program directly reproduces the rich off-shell structure of the Green's functions already known from previous Ising model investigations.[3,4,6]

First, analyticity and crossing state that the matrix elements

$$^{\text{out}}\langle p_1 \cdots p_m|\sigma(0)|p_{m+1}\cdots p_n\rangle^{\text{in}}$$

for fixed $n$ are all given as appropriate boundary values of a single real analytic function $F^{(n)}(\theta_{ij})$ $(1\le i<j\le n)$ totally symmetric in all the $\theta_{ij}$; we are employing covariant state normalization

$$^{\text{in}}\langle p_1' \cdots p_n'|p_1\cdots p_n\rangle^{\text{in}} = \sum_{\text{perm}\,\tau}\prod_{i=1}^{n} 4\pi\delta(\theta_i-\theta'_{\tau(i)})$$

and field normalization $\langle 0|\sigma(0)|p\rangle=1$. Explicitly for for $\theta_r\ne\theta_s$,

$$^{\text{out}}\langle p_1 \cdots p_m|\sigma(0)|p_{m+1}\cdots p_n\rangle^{\text{in}} = F^{(n)}(|\theta_i-\theta_j|;\, i\pi-|\theta_r-\theta_s|;\, |\theta_k-\theta_l|), \quad (6)$$

where

$1\le i<j\le m$, $1\le r\le m<s\le n$, and $m<k<l\le n$.

We can restrict attention to $n$ odd since $F^{(n)}\equiv 0$ for $n$ even due to $Z_2$ conservation. By means of $CPT$ invariance, unitarity, and the no-particle production property, one derives the generalized Watson equations[7] which, by invoking the hypothesis that the $F^{(n)}$ are meromorphic functions, can be written as equations which are valid for all $\theta_{ij}$,

$$F^{(n)}(\theta_{ij};\, i\pi-\theta_{rs};\, \theta_{kl}) = S^{(m)}(\theta_{ij})F^{(n)}(-\theta_{ij};\, i\pi+\theta_{rs};\, -\theta_{kl})S^{(n-m)}(\theta_{kl}). \quad (7)$$

The solution $F^{(n)\min}$ of Eq. (7) with $S^{(n)}$ given by Eq. (5) having no poles and no zeros in the physical strips $0<\operatorname{Im}\theta_{ij}<\pi$ and only a simple zero at $\theta_{ij}=0$ is[8]

$$F^{(n)\min}(\theta_{ij}) = \prod_{1\le i<j\le n}\left(-i\sinh\frac{\theta_{ij}}{2}\right). \quad (8)$$

The desired matrix elements are then given by

$$F^{(n)}(\theta_{ij}) = K^{(n)}(\theta_{ij})F^{(n)\min}(\theta_{ij}), \quad (9)$$

where the function $K^{(n)}$ is a solution of the Watson equations with $S=1$ and contains the poles determined by one-particle intermediate states in subchannels.

Let us first consider the case $n=3$. Since there are no bound states there is only the one-particle pole [e.g. for $m=1$ in Eq. (6)] at $(p_1-p_2-p_3)^2=1$. Hence we obtain for the three-particle form factor

$$F^{(3)}(\theta_{ij}) = \frac{4Z^{(3)}}{\cosh\theta_{12}+\cosh\theta_{13}+\cosh\theta_{23}+1}\,F^{(3)\min}(\theta_{ij}). \quad (10)$$

The constant $Z^{(3)}$ is determined to be $Z^{(3)}=2$ by the requirement that using the expression (10) reproduces the correct $T$-matrix element.

For arbitrary (odd) $n\ge 5$ the arguments are just slightly more involved. Again the functions $K^{(n)}$ must have poles at $\cosh\theta_{ij}+\cosh\theta_{ik}+\cosh\theta_{jk}+1=0$ where any sets of particles $i, j, k$ are combined to give a one-particle intermediate state. However, the residues of "higher poles" where more than three particles build up a one-particle state must vanish, since the presence of such poles would be in contradiction to the absence of particle production. Hence the minimality hypothesis implies that $K^{(n)}$ has in this case the form

$$K^{(n)}(\theta_{ij}) = \frac{R^{(n)}(\theta_{ij})}{\prod_{1\le i<j<k\le n}\frac{1}{4}(\cosh\theta_{ij}+\cosh\theta_{ik}+\cosh\theta_{jk}+1)}, \quad (11)$$

where $R^{(n)}(\theta_{ij})$ has no poles. The function $R^{(n)}$ must, however, possess sufficient zeros to modify the severe singularities of the denominator at points $\theta_r=\theta_s$ in the matrix elements (6) to simple poles. The minimal solution is manifestly

$$R^{(n)}(\theta_{ij}) = Z^{(n)}\prod_{1\le i<j\le n}\left[\tfrac{1}{2}(1+\cosh\theta_{ij})\right]^{(n-3)/2}. \quad (12)$$

Thus we obtain for the matrix elements

$$^{\text{out}}\langle p_1 \cdots p_m|\sigma(0)|p_{m+1}\cdots p_n\rangle^{\text{in}} = Z^{(n)}i^{(n-1)/2}\prod_{1\le i<j\le m}\tanh\frac{|\theta_i-\theta_j|}{2}\prod_{\substack{1\le r\le m\\ m<s\le n}}P\coth\frac{\theta_r-\theta_s}{2}\prod_{m<k<l\le n}\tanh\frac{|\theta_k-\theta_l|}{2}. \quad (13)$$

Note the principal part for the singularity at $\theta_r=\theta_s$. For the $S$ matrix $S=-1$ this is the only distributional character consistent with the Watson equations and general principles, e.g. parity invariance. Finally the constants $Z^{(n)}$ are determined by the conditions that the matrix elements reproduce the correct $S$-matrix elements (5). Employing the Lehmann-Symanzik-Zimmermann (LSZ) formalism and repeated use of the formula

$$\lim_{t\to\pm\infty} e^{i\alpha x t}\,\frac{P}{x} = \pm i\pi\epsilon(\alpha)\delta(x) \quad (14)$$

and the Riemann-Lebesgue lemma yields (with the above state normalization)

$$Z^{(n)}=2^{(n-1)/2}. \tag{15}$$

Having deduced all matrix elements of the $\sigma$ field, we can immediately write down expressions for the Green's functions. For example, the two-point function is

$$\tau(p)=\int d^2x\, e^{ipx}\langle T\sigma(x)\sigma(0)\rangle$$
$$=i\int d\kappa^2 \frac{\rho(\kappa^2)}{p^2-\kappa^2+i\epsilon}$$

with spectral function $\rho$ given by

$$\theta(p^0)\rho(p^2)=\sum_{n\text{ odd}}\frac{1}{n!}\prod_{k=1}^{n}\left(\int\frac{d\theta_k}{4\pi}\right)2\pi\,\delta^{(2)}\left(p-\sum p_k\right)2^{n-1}\prod_{1\le i<j\le n}\tanh^2\frac{\theta_{ij}}{2}. \tag{16}$$

There are similar formulas for the $n$-point Green's functions which are all in agreement with those derived in Refs. 3, 4 and 6. This agreement gives us some confidence in the feasibility of the bootstrap program and also supports the validity of the minimality hypotheses.

The method has indeed been previously applied[7] to matrix elements of the source of the sine-Gordon field between three first breather states and the postulated result was found in agreement in the first orders of perturbation theory. The generalization to the form factors $n>3$ breather states is more complicated than the model considered here. Moreover, to obtain the full Green's functions in this case requires matrix elements of the field for states including an arbitrary number of solitons, in fact, only these matrix elements for regimes of the coupling constant where the breather $b_1$ is no longer in the physical spectrum. This involves the solution of Watson equations involving matrices which have not yet been achieved.

Finally we remark that the two-particle $S$-matrix element (4) is also the result of the analytic continuation of the Zamolodchikov O($N$) nonlinear $\sigma$-model isoscalar amplitude $S_0=N\sigma_1+\sigma_2+\sigma_3$ to $N=1$ (Refs. 9, 10). Using this terminology, it should prove an instructive exercise to carry out the bootstrap program for the supersymmetric O(1) nonlinear $\sigma$ model.[1] The determination of the correlation functions for one of the more involved soliton field theories remains, however, a challenging open problem.

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[1]For recent comprehensive reviews see M. Karowski, Report No. FUB-HEP 78/7, talk presented at the Winter Workshop, Les Houches, 1978, to be published in Phys. Rep.; R. Shankar, report, Yale University, New Haven, Conn., talk given at the APS Meeting at Washington, D. C. (unpublished).

[2]The Ising model in the scaling limit is defined by taking the correlation functions in the limit

$$S(r_1,\cdots,r_n)=\lim\left|1-\frac{T}{T_c}\right|^{-n\beta}\langle\sigma_{R_1}\cdots\sigma_{R_n}\rangle$$

with $R_i\to\infty$ and $r_i=R_i|1-T/T_c|$ fixed. The exact correlation functions are known. See Ref. 3.

[3]B. M. McCoy, C. A. Tracy, and T. T. Wu, Phys. Rev. Lett. 38, 793 (1977).

[4]M. Sato, T. Miwa, and M. Jimbo, report, Kyoto, 1977 (unpublished).

[5]The currents associated with the infinite set of conservation laws are probably complicated when expressed in terms of $\sigma(x)$ but simple in terms of the underlying free Majorana field.

[6]B. Schroer and T. Truong, Phys. Lett. 72B, 371 (1978); 73B, 149 (1978) and Nucl. Phys. B144, 80 (1978).

[7]M. Karowski and P. Weisz, Nucl. Phys. B139, 455 (1978).

[8]The choice $\sinh(\theta/2)$ instead of $[\sinh(\theta/2)]^{-1}$ for bosons is motivated by previous investigations (see Ref. 7) in the sine-Gordon theory.

[9]A. B. Zamolodchikov and Aj. B. Zamolodchikov, Nucl. Phys. B133, 525 (1978).

[10]This has also been observed by E. Brezin and G. Parisi (private communication).