SCATTERING AMPLITUDES OF THE GROSS–NEVEU AND NONLINEAR σ-MODELS IN HIGHER ORDERS OF THE 1/N-EXPANSION

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The exact S-matrices proposed by Alexander and Alexey Zamolodchikov for the nonlinear σ-model and Gross–Neveu model are verified to order $1/N^2$ perturbation theory. This provides a good check of the nature of the bound state spectrum.

The Gross-Neveu (GN) and nonlinear σ-models (NLS) in two dimensions are described by the lagrangians

$$\mathcal{L}^{\mathrm{GN}} = \sum_{j=1}^{N} \bar{\psi}_j \, i\partial\!\!\!/ \psi_j + \tfrac{1}{2} g \left(\sum_{j=1}^{N} \bar{\psi}_j \psi_j \right)^2 ,$$

$$\mathcal{L}^{\mathrm{NLS}} = \tfrac{1}{2} \sum_{j=1}^{N} (\partial_\mu n_j)^2 \quad \text{with} \quad g \sum_{j=1}^{N} n_j^2 = 1 .$$

Exact S-matrices for these models were recently proposed by Zamolodchikov and Zamolodchikov [1,2] who analysed the factorization constraints [3] for the case of scattering of an O(N) N-plet of massive particles. Their arguments for identifying the S-matrices obtained by the factorization condition to those of the models given by $\mathcal{L}^{\mathrm{GN}}$ and $\mathcal{L}^{\mathrm{NLS}}$ relied essentially on a check on lowest order of the $1/N$-expansion. Shortly later it was recognized that the quantum NLS- [4,5] and GN-models [6] possess infinite sets of conservation laws which imply [7] the factorization equations.

In the present note we calculate up to $1/N^2$ the S-matrices of the GN- and NLS-models. Because of the ambiguity in the solution of the factorization equations (which is related to the spectrum), our calculation is a nontrivial check for the correctness of the spectrum of the GN- and NLS-models which is exhibited by the chosen S-matrices. Especially the rich particle spectrum of the GN-model, as determined in the semiclassical approximation [8], is confirmed.

Consider the elastic scattering of an O(N) isovector N-plet of particles P_i of mass m. The S-matrix elements are given by

$$ {}^{\mathrm{out}}\langle P_j(\tilde{p}_1) P_l(\tilde{p}_2) \,|\, P_i(p_1) P_k(p_2) \rangle^{\mathrm{in}} $$
$$ = {}_{ik}S_{jl}(\theta, N)\, \delta(\tilde{p}_1^1 - p_1^1)\, \delta(\tilde{p}_2^1 - p_2^1) \qquad (1) $$
$$ \pm {}_{ik}S_{lj}(\theta, N)\, \delta(\tilde{p}_1^1 - p_2^1)\, \delta(\tilde{p}_2^1 - p_1^1), $$

with

$$ {}_{ik}S_{jl}(\theta, N) = \sigma_1(\theta, N)\, \delta_{ik}\delta_{jl} + \sigma_2(\theta, N)\, \delta_{ij}\delta_{kl} + \sigma_3(\theta, N)\, \delta_{il}\delta_{jk}, $$

where θ the rapidity variable is given by

$$ p_1 p_2 = m^2 \,\mathrm{ch}\, \theta , $$

and the +(−) in (1) refers to bosons (fermions), respectively.

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Fig. 1. Tree graph contribution to σ_2.

For special models, such as the NLS and the GN, the S-matrix factorizes in terms of two-particle scattering matrices and the S-matrix fulfills severe constraints [3]. Indeed, as Zamolodchikov and Zamolodchikov [1] showed, the amplitude σ_3 is simply related to σ_2 (remember crossing: $\sigma_1(\mathrm{i}\pi-\theta)=\sigma_3(\theta)$) by:

$$\sigma_3(\theta,N)=-\frac{2\pi\mathrm{i}}{N-2}\frac{\sigma_2(\theta,N)}{\theta}. \tag{2}$$

And the general solution of σ_2 is given by

$$\sigma_2(\theta,N)=\left[\prod_{k=1}^{L}\frac{\mathrm{sh}\,\theta+\mathrm{i}\sin\alpha_k}{\mathrm{sh}\,\theta-\mathrm{i}\sin\alpha_k}\right]\sigma_2^{(0)}(\theta,N)$$

where the real parameters α_k correspond to poles in the physical plane. The minimal solution is given by

$$\sigma_2^{(0)}(\theta,N)=Q(\theta,N)Q(\mathrm{i}\pi-\theta,N)$$

with

$$Q(\theta,N)=\frac{\Gamma(1/(N-2)-(\mathrm{i}\theta/2\pi))\Gamma(\frac{1}{2}-(\mathrm{i}\theta/2\pi))}{\Gamma(-\mathrm{i}\theta/2\pi)\Gamma(\frac{1}{2}+1/(N-2)-(\mathrm{i}\theta/2\pi))}.$$

For $1/N$-perturbation calculations it is more convenient to cast the solution into the form

$$\ln\sigma_2^{(0)}(\theta,N)=-\int_0^\infty\frac{\mathrm{d}t}{t}\frac{\mathrm{ch}\,\frac{1}{4}t(1+(2\mathrm{i}\theta/\pi))}{\mathrm{ch}\,\frac{1}{4}t}$$

$$\times\{1-\exp(-t/(N-2))\}\quad\text{for } 0<\mathrm{Im}\,\theta<\pi.$$

In the O(N) NLS-model no bound states are expected and, hence,

$$\sigma_2^{\mathrm{NLS}}(\theta,N)=\sigma_2^{(0)}(\theta,N)$$

was proposed [1].

Assuming for the U(N) GN-model the qualitative nature of the rich bound state spectrum which was obtained in the semiclassical analysis [8], the exact S-matrix is proposed [2] to be given by

$$\sigma_2^{\mathrm{GN}}(\theta,2N)=\frac{\mathrm{sh}\,\theta+\mathrm{i}\sin\pi/(N-1)}{\mathrm{sh}\,\theta-\mathrm{i}\sin\pi/(N-1)}\sigma_2^{(0)}(\theta,2N).$$

We expand the amplitudes to order $1/N^2$ and obtain

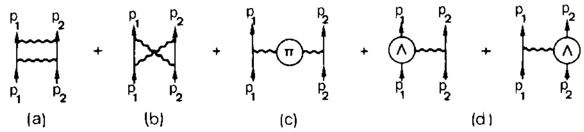

Fig. 2. Contributions in order $1/N^2$ to σ_2.

for the T-matrix elements

$$T^{\mathrm{NLS}}(\theta,N)=4\,\mathrm{sh}\,\theta(\sigma_2^{\mathrm{NLS}}(\theta,N)-1)$$

$$=-\frac{8\pi\mathrm{i}}{N}+\frac{1}{N^2}(\chi(\theta)-16\pi\mathrm{i})+\mathrm{O}(N^{-3}) \tag{3a}$$

$$T^{\mathrm{GN}}(\theta,N)=4\,\mathrm{sh}\,\theta\,(\sigma_2^{\mathrm{GN}}(\theta,2N)-1)$$

$$=\frac{4\pi\mathrm{i}}{N}+\frac{1}{4N^2}(\chi(\theta)+16\pi\mathrm{i})+\mathrm{O}(N^{-3}) \tag{3b}$$

where

$$\chi(\theta)=2\,\mathrm{sh}\,\theta\left[\int_0^\infty\mathrm{d}t\,t\frac{\mathrm{ch}\,\frac{1}{4}t(1+2\mathrm{i}\theta/\pi)}{\mathrm{ch}\,\frac{1}{4}t}-\frac{4\pi^2}{\mathrm{sh}^2\theta}\right]$$

which has the behaviour

$$\chi(\theta)\approx16\pi^2/\theta\qquad\text{as }\theta\to0,$$

at threshold. Since the linearity relation (2), which is a consequence of the conversation laws [4–6], relates σ_3 to σ_2, it is sufficient to calculate T^{GN} and T^{NLS} defined in eq. (3).

To first order $1/N$ only the tree diagram (fig. 1) contributes and one obtains:

$$T_{\mathrm{tree}}^{\mathrm{NLS}}(\theta,N)=-(1/N)\,8\pi\mathrm{i}$$

$$T_{\mathrm{tree}}^{\mathrm{GN}}(\theta,N)=(1/N)\,4\pi\mathrm{i}$$

in agreement with (3).

In second order $1/N^2$ a variety of graphs contribute (fig. 2). Of these only the box diagrams 2(a) and 2(b) give energy dependent contributions. Due to the asymptotic $(\log k^2)^{-1}$ behaviour of the propagator

$$D^{\mathrm{GN}}(k^2)=-\frac{2\pi\mathrm{i}}{N}\frac{\mathrm{th}\,\frac{1}{2}\phi}{\phi}\quad\text{where } k^2=-4m^2\,\mathrm{sh}^2\,\tfrac{1}{2}\phi$$

$T_{\mathrm{Box}}^{\mathrm{GN}}(\theta,N)$ is convergent. $T_{\mathrm{Box}}^{\mathrm{NLS}}(\theta,N)$ diverges as the ultraviolet cut-off parameter $\Lambda\to\infty$. But again due to $(\log k^2)^{-1}$ factor in

$$D^{\mathrm{NLS}}(k^2) = \frac{8\pi \mathrm{i}}{N} m^2 \frac{\mathrm{sh}\,\phi}{\phi}$$

it is sufficient to make only one subtraction. First we can show

$$T^{\mathrm{NLS}}_{\mathrm{Box}} - 4T^{\mathrm{GN}}_{\mathrm{Box}} = \text{const.}$$

Hence, it is sufficient to check only T^{GN} in detail to obtain agreement for T^{NLS} up to a constant. We calculate $T^{\mathrm{GN}}_{\mathrm{Box}}(\theta, N)$ by introducing the dispersion relation

$$[D^{\mathrm{GN}}(k^2)]^2 = \frac{(2\pi)^2}{N^2} \int_{-\infty}^{\infty} \mathrm{d}\phi \left[\frac{\phi}{(\phi^2+\pi^2)^2} \frac{\mathrm{ch}^3 \frac{1}{2}\phi}{\mathrm{sh}\frac{1}{2}\phi} + \frac{1}{\pi^2}\delta(\phi)\right] \frac{4m^2}{k^2 - 4m^2 \mathrm{ch}^2 \frac{1}{2}\phi + \mathrm{i}\epsilon}$$

and then performing the k-integration. We find

$$T^{\mathrm{GN}}_{\mathrm{Box}}(\theta, N) = \frac{1}{N^2}\left\{\tfrac{1}{4}\chi(\theta) - 16\mathrm{i}\left[\frac{1}{\pi^2}\ln 2 + \int_0^{\infty} \mathrm{d}\phi \frac{\phi}{(\phi^2+\pi^2)^2} \mathrm{ch}^2 \tfrac{1}{2}\phi \times (2\,\mathrm{cth}\,\tfrac{1}{2}\phi \ln(2\,\mathrm{ch}\,\tfrac{1}{2}\phi) - \phi)\right]\right\} + O(N^{-3}) \tag{4}$$

reproducing the energy dependent term in (3b).

Finally we evaluate the constant contribution coming from diagrams (2c) and (2d). They are separately divergent but their sum is convergent:

$$T^{\mathrm{GN}}_{2c+2d}(\theta, N) = \frac{8\pi\mathrm{i}}{N^2}\left\{1 + \int_0^{\infty} \mathrm{d}\phi \frac{\mathrm{cth}\,\frac{1}{2}\phi}{\phi^2+\pi^2} \times [\tfrac{1}{2}\phi - \mathrm{cth}\,\tfrac{1}{2}\phi \ln \mathrm{ch}\,\tfrac{1}{2}\phi]\right\} + O(N^{-3}). \tag{5}$$

The final contribution comes from the (finite) Z_2^2 factor multiplying the one-particle irreducible 4-point function

$$(Z_2^2 - 1)T^{\mathrm{GN}}_{\mathrm{tree}}(\theta, N) = -\frac{16\pi\mathrm{i}}{N^2}\left\{\frac{1}{4} + \int_0^{\infty} \mathrm{d}\phi \frac{\mathrm{ch}^2 \frac{1}{2}\phi}{\phi^2+\pi^2} \times [\tfrac{1}{2}\mathrm{cth}\,\phi - \ln(2\,\mathrm{ch}\,\tfrac{1}{2}\phi)]\right\} + O\left(\frac{1}{N^3}\right). \tag{6}$$

The ϕ-integrations can be done by means of Laplace transformations.

Summing up the contributions (4), (5) and (6) we reproduce the Zamolodchikov prediction. Details of the present investigation [9] and the calculation of the form factors [10] will be published elsewhere.

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