Higher Order $1/N$ Calculations in the Gross-Neveu and Nonlinear σ models

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Abstract:

The exact S-matrices proposed by Alexander and Alexey Zamolodchikov for the nonlinear σ-model and the Gross-Neveu model are verified to order N^{-2} perturbation theory. This provides a good check of the nature of the Gross-Neveu model bound state spectrum derived by Dashen, Hasslacher and Neveu in the semiclassical approximation. The behavior of the σ-propagator in the Gross-Neveu model is investigated.

Contents

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1 Introduction

The Gross-Neveu (GN) [1] and nonlinear σ-models [2, 3] in two dimensions described by the Lagrangians

$$\mathcal{L}^{GN} = \sum_{\alpha=1}^{N} \bar{\psi}_\alpha i\gamma\partial\psi_\alpha + \frac{1}{2}g\left(\sum_{\alpha=1}^{N}\bar{\psi}_\alpha\psi_\alpha\right)^2$$

$$\mathcal{L}^{NLS} = \frac{1}{2}\sum_{i=1}^{N}(\partial_\mu n_i)^2 \quad \text{with } g\sum_{i=1}^{N} n_i^2 = 1$$

are interesting models in their own right. The GN model is a prototype for the phenomenon of dynamical symmetry breaking and the NLS model a prototype in which the interaction originates entirely from constraints on the allowed field configurations. The study of these models (and generalizations thereof [4]) is further motivated by the fact that they share certain properties in common with the physically relevant non.abelian gauge theories. To gain insight in these particular aspects it is useful to study them first in simpler non-gauge theories in two dimensions. The important common property, That was indeed one of the main motivations of the pioneering work of Gross and Neveu, is that the models are scale invariant at the classical level and as quantum field theories are asymptotically free [1, 2]. The analogies, however, extend somewhat deeper than at first anticipated: all the models share features in common with completely integrable systems. The GN and the NLS models posses infinite sets of conservation laws and it is possible that this is also the case for pure gauge theories[1].

The existence of an infinite set of conservation laws in the classical nonlinear σ-model was first discussed by Pohlmeyer [3]. The isospin invariant conserved currents of Pohlmeyer are rather singular objects, rational functions of the fields and their derivatives, redering the associated classical charges illdefined. In the quantum theory, however, due to the anomalies resulting in the non-vanishing of the trace of the energy momentum tensor, the conservation laws corresponding to the Pohlmeyer currents become respectable. This startling phenomenon was pointed out by Polyakov [7] who picturesquely called it the rehabilitation of conservation laws. Another complementary approach to the the problem developed simultaneously was indicated by Lüscher and Pohlmeyer [8] who constructed a new set of well defined conserved non-local charges in the classical model. Subsequently

[1]The $O(3)$ σ-model shares the additional property of having exact multi-instanton [5] and meron solutions [6]

the highly non-trivial step of formulating the corresponding non-local charges in the quantum theory was completed by Lüscher [9], who has also constructed analogous charges in the Gross-Neveu model [10]. Non-local charges are of conceptional importance because they may have an analogue in higher dimensions.

Among the physical consequences of the conservation laws, in both formulations, are the conservation of the set of momenta [11] in a scattering process and factorisation of the S-matrix [12]. Factorisation and T-invariance, in turn, imply non-trivial constraints on the 2-particle S-matrix elements [13]. The "minimal S-matrix" with the minimal number of poles and zeros in the physical sheet is uniquely determined. the general solution is given in terms of these and factors of particular structure including the possibility of introducing poles in the physical sheet. Zamolodchikov and Zamolodchikov [14] analyzed the factorisation constraints for the case of scattering of an $O(N)$ N-plet of massive particles, relevant for the GN and NLS models. Assuming no bound states occur in the NLS model and assuming the qualitative nature of the semiclassical analysis of Dashen, Hasslacher and Neveu [15] Zamolodchikov and Zamolodchikov [14, 16] speculated on the exact form of the S-matrix for both models. These results are summarized in section 2.

In this paper we perform a check of the proposed S-matrices to order $1/N^2$ perturbation theory. To establish notation and for completeness the Feynman rules are discussed in Appendix A together with a discussion of the renormalization theory in the GN model up to order $1/N^2$. We will return to a more complete discussion of the renormalization theory and related matters in another paper. The only divergent renormalization parameter is the bare coupling. This means that, introducing an ultraviolet cutoff Λ, it is sufficient to perform only an infinite renormalization of the dynamically generated mass to overcome the $\ln\Lambda$ and $\ln\ln\Lambda$ divergences encountered at intermediate stages. The GN model has then only the parameters m, the physical ψ-mass and N [1]. this situation also pertains in the NLS model, the renormalization of which has been shown by Brezin et al [17] in $2+\varepsilon$ dimensions ($\varepsilon \geq 0$) and recently been analyzed by Symanzik [18] directly in the $1/N$-expansion.

In section 3 we discuss the σ-propagator in the GN model, the σ-self-energy (at zero momentum) being required in our analysis. A previous investigation was performed by Schonfeld [19] who claimed that the mass of the first isoscalar bound state deviates from $2m$ only in order $1/N^2$. This result was not only very difficult to reconcile with the analysis of Dashen, Hasslacher and Neveu [15] but is in conflict with the proposed S-matrix of Zamolodchikov and Zamolodchikov [16]. We disagree with the analysis of Schonfeld at important points. Firstly we claim that the higher order corrections to the self-energy are singular at threshold, thus preventing the determination of the σ-mass from perturbation theoretic calculation of the σ-propagator. Secondly we disagree on the value of the self-energy in leading order at threshold (some of the details are relegated to Appendix B). The overall picture is then in agreement with the quasiclassical analysis.

In the final section 4 we complete the check of the proposed S-matrices [14, 16] to order $1/N^2$ and obtain full agreement. Some of the details are relegated to Appendix C. Having thus obtained confidence in the form of the exact S-matrices in the GN and the NLS models we can go to the next step of examining form factors; this is the topic of a subsequent paper [20].

2 The Exact S-Matrices

First consider the elastic scattering of an $O(N)$ isovector N-plet of particles P_i of mass m. The S-matrix is given by

$$\begin{aligned} & {}^{out}\langle\, P_j(\tilde{p}_1)P_l(\tilde{p}_2)\,|\,P_i(p_1)P_k(p_2)\,\rangle^{in} \\ & = {}_{ik}S_{jl}(\theta,N)\delta(\tilde{p}_1-p_1)\delta(\tilde{p}_2-p_2) \pm {}_{ik}S_{jl}(\theta,N)\delta(\tilde{p}_1-p_2)\delta(\tilde{p}_2-p_1) \end{aligned} \tag{2.1}$$

with

$${}_{ik}S_{jl}(\theta,N) = \sigma_1(\theta,N)\delta_{ik}\delta_{jl} + \sigma_2(\theta,N)\delta_{ij}\delta_{kl} + \sigma_3(\theta,N)\delta_{il}\delta_{kj} \tag{2.2}$$

where θ the rapidity variable is given by

$$p_1p_2 = m^2\cosh\theta \tag{2.3}$$

and the $+(-)$ in (2.1) refers to bosons (fermions), respectively,. Crossing relates

$$\begin{aligned} \sigma_3(\theta,N) &= \sigma_1(i\pi-\theta,N) \\ \sigma_2(\theta,N) &= \sigma_2(i\pi-\theta,N)\,. \end{aligned} \tag{2.4}$$

The invariant s-channel amplitudes A_0, A_+, A_- corresponding to scalar, traceless symmetric and antisymmetric representations of $O(N)$, respectively, are given by

$$\begin{aligned} A_0 &= N\sigma_1 + \sigma_2 + \sigma_3 \\ A_+ &= \sigma_2 + \sigma_3 \\ A_- &= \sigma_2 - \sigma_3\,. \end{aligned} \tag{2.5}$$

In terms of these elastic unitarity simply becomes

$$A_0(\theta,N)A_0(-\theta,N) = A_+(\theta,N)A_+(-\theta,N) = A_-(\theta,N)A_-(-\theta,N) = 1\,. \tag{2.6}$$

For special models such as the NLS and GN the absence of particle production and factorization impose severe restrictions on the form of the S-matrix [13]. Indeed, as Zamolodchikov and Zamolodchikov [14] showed, the amplitude σ_3 is necessarily related to σ_2 as follows

$$\sigma_3(\theta,N) = -\frac{2\pi i}{N-2}\frac{\sigma_2(\theta,N)}{\theta}\,. \tag{2.7}$$

The crossing symmetric amplitude σ_2 now satisfies the equation

$$\sigma_2(\theta,N)\sigma_2(-\theta,N) = \frac{\theta^2}{\theta^2 + 2\pi^2/(N-2)^2} \tag{2.8}$$

the general solution of which, having singularities on the imaginary axis, has the form

$$\sigma_2(\theta,N) = \prod_{k=1}^{L}\frac{\sinh\theta + i\sin\alpha_k}{\sinh\theta - i\sin\alpha_k}\sigma_2^{(0)}(\theta,N) \tag{2.9}$$

where α_k are real and $\sigma_2^{(0)}$ is the solution with no poles ore zeros in the physical strip $0 < \operatorname{Im}\theta < \pi$

$$\sigma_2^{(0)}(\theta, N) = Q(\theta, N)Q(i\pi - \theta, N) \tag{2.10}$$

with

$$Q(\theta, N) = \frac{\Gamma\left(\frac{1}{N-2} - \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right)}{\Gamma\left(-\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} + \frac{1}{N-2} - \frac{i\theta}{2\pi}\right)}.$$

For $1/N$-perturbation calculations it is usually more convenient to cast the solution into the form

$$\ln\sigma_2^{(0)}(\theta, N) = -\int_0^\infty \frac{dt}{t}\frac{\cosh\frac{t}{4}\left(1 + 2i\theta/\pi\right)}{\cosh\frac{t}{4}}\left(1 - e^{-\frac{t}{N-2}}\right) \tag{2.11}$$

valid for $\pi > \operatorname{Im}\theta > 0$.

In the $O(N)$ NLS model no bound states are expected and, hence, Zamolodchikov and Zamolodchikov [14] proposed

$$\sigma_2^{NLS}(\theta, N) = \sigma_2^{(0)}(\theta, N). \tag{2.12}$$

For the $U(N)$ Gross-Neveu model, on the other hand, the quasi-classical considerations of Dashen, Hasslacher, and Neveu [15] suggest a rather rich bound state spectrum.

The levels are given by a principal quantum number n

$$m_n = m\frac{\sin\frac{1}{2}n\alpha(N)}{\sin\frac{1}{2}\alpha(N)} \tag{2.13}$$

and at each level n the particles occur in multiplets corresponding to $O(2N)$ totally antisymmetric tensors of rank $n, n-2, n-4, \ldots$. To reproduce the energy levels it is sufficient to include only one factor ($L = 1$) in (2.9). Then the spectrum (2.13) follows from factorization [21] and a bound state with mass m_n can be considered as a bound state of n lowest level particles. The spectrum of this model presumably contains also the so-called Callan, Coleman, Gross, and Zee kinks [15] which cannot be thought of as bound states of any finite number of "elementary" particles. Our present considerations yield no extra information on these. The expression for the function $\alpha(N)$ is determined if one imposes the qualitative degeneracy of the semiclassical spectrum. This is achieved if only the invariant amplitudes $A_0(\theta, 2N)$ and $A_-(\theta, 2N)$ have poles at $\theta = i\alpha$ and $A_+(\theta, 2N)$ is regular there. Now

$$A_\pm(\theta, 2N) = \left(1 \mp \frac{i\pi}{N-1}\frac{1}{\theta}\right)\sigma_2(\theta, 2N) \tag{2.14}$$

and

$$A_0(\theta, 2N) = -\frac{i\pi + \theta}{i\pi + \theta}A_-(\theta, 2N)$$

and, hence, it follows

$$\alpha(N) = \frac{\pi}{N-1}, \quad (n = 1, \ldots, N-1). \tag{2.15}$$

It is remarkable that this is exactly th4 value of $\alpha(N)$ suggested by Dashen, Hasslacher, and Neveu from apparently disconnected arguments involving analogies with the Sine-Gordon Theory.

To summarize, the exact form of the amplitude σ_2 for the Gross-Neveu model is proposed, again by Zamolodchikov and Zamolodchikov [16] to be given by

$$\sigma_2^{GN}(\theta, 2N) = \frac{\sinh + i \sin \pi/(N-1)}{\sinh - i \sin \pi/(N-1)} \sigma_2^{(0)}(\theta, 2N) . \tag{2.16}$$

Although the Gross-Neveu model has full $O(2N)$ symmetry it is for various calculations often convenient to use only the $U(N)$ symmetry in which pairs of self-conjugate particles are (arbitrarily) combined into a particle P_α and antiparticle A_α, such that the particles belong to the fundamental representation of $U(N)$. The general elastic scattering elements are given by (assuming fermions)

$$\begin{aligned} &{}^{out}\langle\, P_\beta(\tilde{p}_1) A_\delta(\tilde{p}_2) \,|\, P_\alpha(p_1) A_\gamma(p_2) \,\rangle^{in} \\ &\qquad = {}_{\alpha\gamma}F_{\beta\delta}(\theta, N) \delta(\tilde{p}_1 - p_1) \delta(\tilde{p}_2 - p_2) - {}_{\alpha\gamma}B_{\beta\delta}(\theta, N) \delta(\tilde{p}_1 - p_2) \delta(\tilde{p}_2 - p_1) \end{aligned} \tag{2.17}$$

$$\begin{aligned} &{}^{out}\langle\, P_\beta(\tilde{p}_1) P_\delta(\tilde{p}_2) \,|\, P_\alpha(p_1) P_\gamma(p_2) \,\rangle^{in} \\ &\qquad = {}_{\alpha\gamma}S_{\beta\delta}(\theta, N) \delta(\tilde{p}_1 - p_1) \delta(\tilde{p}_2 - p_2) - {}_{\alpha\gamma}S_{\beta\delta}(\theta, N) \delta(\tilde{p}_1 - p_2) \delta(\tilde{p}_2 - p_1) \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} {}_{\alpha\gamma}F_{\beta\delta}(\theta, N) &= t_1(\theta, N) \delta_{\alpha\beta}\delta_{\gamma\delta} + t_2(\theta, N) \delta_{\alpha\gamma}\delta_{\beta\delta} \\ {}_{\alpha\gamma}B_{\beta\delta}(\theta, N) &= r_1(\theta, N) \delta_{\alpha\beta}\delta_{\gamma\delta} + r_2(\theta, N) \delta_{\alpha\gamma}\delta_{\beta\delta} \\ {}_{\alpha\gamma}S_{\beta\delta}(\theta, N) &= u_1(\theta, N) \delta_{\alpha\beta}\delta_{\gamma\delta} + u_2(\theta, N) \delta_{\alpha\gamma}\delta_{\beta\delta} \,. \end{aligned}$$

Crossing enforces the relations

$$\begin{aligned} t_1(\theta, N) &= u_1(i\pi - \theta, N) \\ t_2(\theta, N) &= u_2(i\pi - \theta, N) \\ r_1(\theta, N) &= r_2(i\pi - \theta, N) \,. \end{aligned}$$

If factorization and full $O(2N)$ symmetry pertains, one can identify

$$\begin{aligned} u_1(\theta, N) &= t_1(\theta, N) = \sigma_2(\theta, N) \\ u_2(\theta, N) &= r_1(\theta, N) = \sigma_3(\theta, N) \,. \end{aligned}$$

Finally we expand the proposed exact amplitudes to order $1/N^2$ and thus obtain for the T-matrix elements

$$\begin{aligned} T^{NLS}(\theta, N) &\equiv 4 \sinh\theta \left(\sigma_2^{NLS}(\theta, N) - 1 \right) = -\frac{1}{N} 8\pi i + \frac{1}{N^2} \left(\chi(\theta) - 16\pi i \right) + O(1/N^3) \\ T^{GN}(\theta, N) &\equiv 4 \sinh\theta \left(u_1^{GN}(\theta, N) - 1 \right) = \frac{1}{N} 4\pi i + \frac{1}{4N^2} \left(\chi(\theta) + 16\pi i \right) + O(1/N^3) \end{aligned} \tag{2.19}$$

with

$$\chi(\theta) = 2\sinh\theta \left(\int_0^\infty dt\, t \frac{\cosh\frac{1}{4}t\,(1+2i\theta/\pi)}{\cosh\frac{1}{4}t} - \frac{4\pi^2}{\sinh^2\theta} \right)$$

which has the following singular behaviour at threshold

$$\chi(\theta) \approx \frac{16\pi^2}{\theta} \quad \text{as } \theta \to 0\,.$$

It is these expressions that we wish to check, and this we do in section 4. In the next section we first discuss the σ-propagator in the Gross-Neveu model not only because we need some information concerning it but also to correct some errors in previous literature [19].

3 The σ-Propagator

In this section we calculate the full σ-propagator $D'(k^2)$ up to order $1/N^2$. In lowest order (i.e. $1/N$) $D'^{\,-1} = D^{-1}$ is given by a subtracted fermion loop with two lines, c.f. Appendix A and Fig. 1.

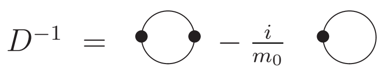

Figure 1: Diagrams contributing to the bare inverse σ-propagator

$$D(k^2, m) = -\frac{2\pi i}{N}\frac{\tanh\phi/2}{\phi} \tag{3.1}$$

with $k^2 = -4m^2\sinh^2\phi/2$. The function $D(k^2)$ is analytic in the physical k^2-plane with a branch cut from $4m^2$ to $+\infty$. At threshold $k^2 \to 4m^2$, i.e. $\hat{\phi} = i\pi - \phi \to 0$ the inverse propagator vanishes like

$$D^{-1} \approx \frac{N}{4}\hat{\phi} \tag{3.2}$$

corresponding to $D^{-1} \approx \sqrt{4m^2 - k^2}$, the threshold behaviour of a free fermion loop.

The full σ-propagator

$$D'(k^2) = \left(D^{-1}(k^2, m_0) - \Pi(k^2)\right)^{-1} \tag{3.3}$$

where $\Pi(k^2)$ is the σ-self-energy will have the following three properties:

i) for large N: $D'(k^2) \to D(k^2)$, since the theory becomes free.

ii) An isoscala fermion-antifermion bound state with mass $m_\sigma = 2m\cos\frac{1}{2}\alpha$ should appear as a pole of D' in the physical plane, i.e. at $\hat{\phi} = i\pi - \phi = i\alpha$ with α pure real and $0 < \alpha < \pi$. Because of i) such poles as functions of N disappear from the physical plane through the threshold for large N.

iii) For generic N (i.e. except for those values of N where possible bound states[2] disappear at threshold) in a neighborhood of $k^2 = 4m^2$ (i.e. $\hat{\phi} = 0$) the threshold behaviour will be determined by the free fermion loop, i.e

$$D'^{-1} \approx const.\,\hat{\phi} \quad \text{for } \hat{\phi} \to 0\,.$$

We shall not prove these properties rigorously but they are natural from WKB-considerations [15] if the $1/N$ expansion converges and they are supported by a form factor calculation [20] in the Gross-Neveu model. In lowest order, i.e. $O(1)$, the Feynman diagrams contributing to $\Pi(k^2)$ are shows in Fig. 2. The calculation can be found in

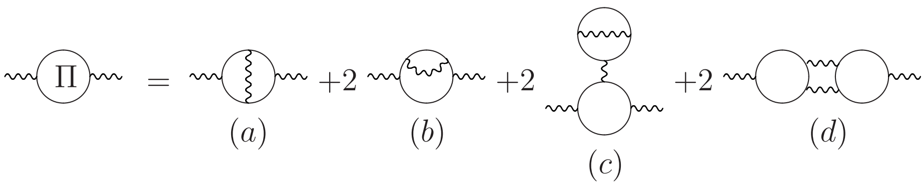

Figure 2: $O(1)$ σ-self-energy diagrams

Appendix B. The only divergent part in eq. 3.3 is proportional to $D^{-1}(k^2, m)$ which means that D^{-1} is "multiplicatively renormalizable"

$$D'(k^2) = Z_3 \left(D^{-1}(k^2, m) - \Pi_c(k^2)\right)^{-1} \tag{3.4}$$

with

$$Z_3 - 1 \approx -2 \int^{\Lambda} \frac{d^2k}{(2\pi)^2} \frac{D(k^2)}{k^2 - 4m_0^2} + O(N^{-2}) \approx \frac{1}{N} \ln\ln\frac{\Lambda}{m} + O(N^{-2})\,.$$

One could ask the question whether it is possible to determine the mass m_σ for a bound state (which approaches the threshold for large N) in some approximation from a $1/N$ perturbation calculation of the σ-self-energy $\Pi(k^2)$. The answer is negative in general.

Because of i) and iii) the inverse σ-propagator behaves like

$$\begin{aligned} D'^{\,-1} &\sim \hat{\phi} && \text{at } \hat{\phi} \approx 0 \\ D'^{\,-1} &\sim \hat{\phi} - i\alpha && \text{at } \hat{\phi} \approx i\alpha = O(1/N)\,. \end{aligned}$$

But a behaviour like $D'^{\,-1} \sim \hat{\phi}\,\left(\hat{\phi} - i\alpha\right)$ in a neighborhood of $\hat{\phi} \approx 0$ (uniformly in N) is in contradiction to i), since for $N \to \infty$ there would be a double zero of $D'^{\,-1}$ at $\hat{\phi} \approx 0$. This must be compensated by a pole at some $i\tilde{\alpha}$ of order $1/N$

$$D'^{\,-1} \sim \hat{\phi}\frac{\hat{\phi} - i\alpha}{\hat{\phi} - i\tilde{\alpha}}\,.$$

In lowest order of $1/N$ expansion this implies

$$D'^{\,-1} \sim \hat{\phi} - i\,(\alpha - \tilde{\alpha}) + O(1/N^2)\,.$$

The zero of $D'^{\,-1}$ in this approximation no longer reflects the position of the true zero, since the $O(1/N^2)$ are singular at $\hat{\phi} = 0$.

[2] In $1/N$ expansion one cannot see such bound states and they are not expected from WKB approximation [15].

The calculations in Appendix B give $\Pi_c(k^2 = 4m^2) = \frac{1}{2}i\pi + O(1/N)$ and with eqs. (3.4,3.2) we obtain

$$D'^{\,-1} = \frac{N}{4}\hat{\phi}\left(1 + O(\hat{\phi})\right) - \frac{1}{2}i\pi\left(1 + O(\hat{\phi})\right) + O(1/N)\,.$$

This result is consistent with Zamolochikov's exact value, discussed in the previous section (see eq. (2.15)), if we assume not only a pole of the σ-propagator at $\hat{\phi} = i\alpha$ but also a zero in the second sheet at $\hat{\phi} = i\tilde{\alpha}$, c.f. Fig. 3.

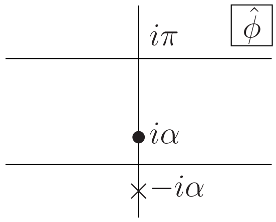

Figure 3: Proposed analytic behaviour of the σ-propagator near threshold

The value $\Pi_c(4m^2) \neq 0$ is not in agreement with a previous investigation [19] of this problem. The result of Appendix C of the latter paper is consistent with with our property iii) for the exact σ-propagator but the proof is wrong.

The property is not true in finite orders of the $1/N$-expansion, since the vertex correction (c.f. Appendix A and Fig. 4) is singular at threshold in finite order. This singularity is also responsible for the non-vanishing value of $\Pi_c(4m^2)$.

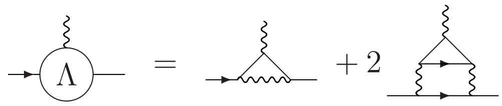

Figure 4: $O(1/N)$ diagrams contributing to the $\sigma\bar{\psi}\psi$ vertex

4 The S-Matrix in Perturbation Theory

In this section we the proposed exact T-matrix elements of the GN and NLS models to order $1/N^2$. We calculate only T^{GN} and T^{NLS} defined in (2.12). This is sufficient to the extend that the linearity relations (2.7) are established for these models. To first order $1/N$ only the tree diagrams Fig. 5 contribute and one obtains, using the Feynman rules listed in Appendix A,

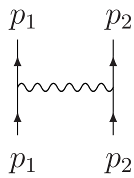

Figure 5: Tree diagrams contributing to the amplitude u_2

$$\begin{aligned} T^{\rm NLS}_{\rm tree}(\theta, N) &= \frac{4}{m^2}\left(\frac{-i}{2}\right)^2 D_\omega(0) = -\frac{1}{N}8\pi i \\ T^{\rm GN}_{\rm tree}(\theta, N) &= \frac{1}{m^2}\bar{u}(p_1)u(p_1)\bar{u}(p_2)u(p_2)(-i)^2 D(0) = \frac{1}{N}4\pi i \end{aligned} \tag{4.1}$$

in agreement with (2.19). In second order $1/N^2$ a variety of graphs contribute and these are depicted in Fig. 6. Of these only the box diagrams 6(a) and 6(b) give energy dependent contributions to the T-matrix elements under considerations. Due to the asymptotic $(\ln k^2)^{-1}$ behaviour of $D(k^2)$, $T^{GN}_{Box}(\theta, N)$ is in fact ("just") convergent. $T^{NLS}_{tree}(\theta, N)$ on the other hand, diverges as the ultraviolet cut-off parameter $\Lambda \to \infty$ but again due to the extra $(\ln k^2)^{-1}$ factor in ${}^2D_\omega(k^2)$ it is sufficient to make only one subtraction. First we show (for all details consult Appendix C) that

$$T^{\rm NLS}_{\rm Box}(\theta, N) - 4T^{\rm GN}_{\rm Box}(\theta, N) = const.$$

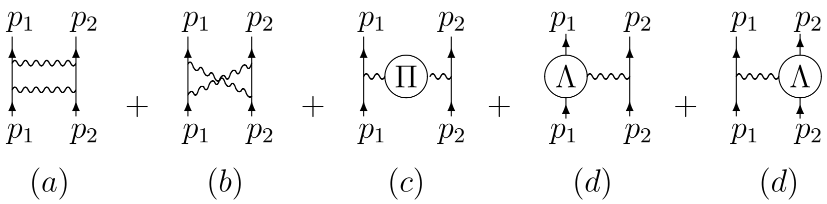

Figure 6: $O(N^{-2})$ diagrams contributing to the amplitude u_2

consistent with the Zamolodchikov prediction (2.19) and, hence, it is sufficient to check only $T^{GN}(\theta, N)$ in detail to obtain agreement for $T^{NLS}(\theta, N)$ up to a constant.

The discontinuity of $T^{GN}(\theta, N)$ across the cut is, of course, determined to order $1/N^2$ by unitarity and the tree diagram (4.1)

$$T^{\rm GN}(\theta,N) - T^{\rm GN}(-\theta,N) = -\frac{4\pi}{N^2\sinh\theta}\left(1+\frac{\sinh^2\theta}{\theta^2}\right) + O(N^{-3}) \tag{4.2}$$

What remains to be checked is the threshold behaviour and the constant term in order $1/N^2$. We first calculate $T^{GN}_{Box}(\theta, N)$ by introducing a dispersion relation for $D^2(k^2)$ and performing the k-integration using the cutting rule (B.3). We find

$$\begin{aligned} T^{\rm GN}_{\rm Box}(\theta,N) &= \frac{1}{N^2}\Bigg\{\frac{1}{4}\chi(\theta) - 16\pi i \\ &\times \left(\frac{1}{\pi^2}\ln 2 + \int_0^\infty d\phi \frac{\phi}{(\phi^2+\pi^2)^2}\cosh^2\frac{\phi}{2}\left(\ln\left(2\coth\frac{\phi}{2}\right)-\phi\right)\right)\Bigg\} + O(N^{-3}) \end{aligned} \tag{4.3}$$

indeed reproducing the precise energy dependent term in (2.19).

Finally we must evaluate the constant contributions coming from diagrams (6c) and (6d). The evaluation of $\Pi(0)$ is straightforward and summing the contributions (B.2)at $k^2 = 0$ we obtain

$$\Pi(0) = \frac{i}{\pi}\left(1 - 2\int^\Lambda \frac{d^2 l}{(2\pi)^2} D^2(l^2, m)\frac{N}{l^2-4m^2}\right) + O(N^{-1}). \tag{4.4}$$

The diagrams (4c) and (4b), on the other hand, yield at $k^2 = 0$

$$\begin{aligned} \Lambda(0,p) &= -i\int^\Lambda \frac{d^2 l}{(2\pi)^2}\Bigg\{\frac{D(l^2,m)}{(\gamma p + \gamma l - m)^2} - 2\frac{D^2(l^2,m)}{\gamma p+\gamma l - m} \\ &\qquad \times \int^\Lambda \frac{d^2 k}{(2\pi)^2}\frac{\mathrm{tr}\,(\gamma k + m)^2(\gamma k + \gamma m + m)}{(k^2-m^2)\left((k+l)^2 - m\right)}\Bigg\} + O(N^{-2}) \\ &= -i\int^\Lambda \frac{d^2 l}{(2\pi)^2}\frac{D(l^2,m)}{\gamma p + \gamma l - m}\left(\frac{1}{\gamma p+\gamma l - m} + \frac{4m}{l^2-4m^2}\right) + O(N^{-2}) \end{aligned} \tag{4.5}$$

The contribution from diagrams (6c) and (6d) are separately divergent but, as discussed in Appendix A, their sum is convergent:

$$\begin{aligned}T^{\rm GN}_{6c+6d}(\theta,N) &= \frac{(-i)^2}{m^2}\bar{u}(p_1)u(p_1)\bar{u}(p_2)u(p_2)\Pi(0)D(0,m) \\ &\qquad + \frac{2(-i)}{m^2}\bar{u}(p_1)u(p_1)\bar{u}(p_2)\Lambda(0,p_2)u(p_2)D(0,m) \\ &= \frac{4\pi i}{N^2}\left\{1+2N\int\frac{d^2l}{(2\pi)^2}D(l^2,m)\left(\frac{4m^2-l^2}{(l^2+2pl)^2}+\frac{1}{l^2-4m^2}\right)\right\}+O(N^{-3}) \\ &= \frac{8\pi i}{N^2}\left\{1+\int_0^\infty d\phi\frac{\coth\phi/2}{\phi^2+\pi^2}\left(\frac{\phi}{2}-\coth\frac{\phi}{2}\ln\coth\frac{\phi}{2}\right)\right\}+O(N^{-3}) \qquad (4.6)\end{aligned}$$

The final contribution comes from the Z_2^2 factor multiplying the one-particle irreducible 4-point function. To order N^{-2} only the tree graph is relevant and one obtains an extra contribution to $T^{GN}(\theta,N)$:

$$\begin{aligned}&\left(Z_2^2-1\right)T^{\rm GN}_{\rm tree}(\theta,N) \\ &= -\frac{16\pi i}{N^2}\left\{\frac{1}{4}+\int_0^\infty d\phi\frac{\coth^2\phi/2}{\phi^2+\pi^2}\left(\frac{\phi}{2}\coth\phi-\ln\left(2\coth\frac{\phi}{2}\right)\right)\right\}+O(N^{-3}) \qquad (4.7)\end{aligned}$$

Summing the contribution (4.3), (4.6), and (4.7) we reproduce the precise Zamolodchikov prediction for the Gross-Neveu model in (2.19). This constitutes a good test of the nature of the semiclassical considerations of Dashen, Haslacher, and Neveu.

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A $1/N$ expansion

In this appendix we give a short review of the derivation of the Feynman rules and the renormalization procedure (up to order $1/N^2$) for the Gross-Neveu model in the $1/N$ expansion. For completeness we write down the Feynman rules of the nonlinear σ-model at the end of this appendix.

The Green's functions of the Gross-Neveu model are given by

$$\langle\, T\,\psi_{A_1}(x_1)\ldots\bar{\psi}_{A_n}(x_n)\,\rangle = Z_2^{-n/2}i^{-n}\frac{\delta}{\delta\bar{\xi}_{A_1}(x_1)}\ldots Z(\xi,\bar{\xi})\ldots\frac{\overleftarrow{\delta}}{\delta\xi_{A_n}(x_n)} \qquad (A.1)$$

where A_i are combined Dirac and $U(N)$ labels and $Z(\xi,\bar{\xi})$ is the Feynman path integral

$$Z(\xi,\bar{\xi}) = \int d\bar{\hat{\psi}}d\hat{\psi}\exp i\left(\mathcal{A}\left(\bar{\hat{\psi}},\hat{\psi}\right)+\int d^2x\left(\bar{\xi}\hat{\psi}+\bar{\hat{\psi}}\xi\right)\right) \qquad (A.2)$$

up to a normalization constant such that $Z(0,0) = 1$. The action is

$$\mathcal{A}\left(\bar{\hat{\psi}},\hat{\psi}\right) = \int d^2x \left(\bar{\hat{\psi}} i\gamma\partial\hat{\psi} + \frac{1}{2}\hat{g}^2\left(\bar{\hat{\psi}}\hat{\psi}\right)^2\right) \tag{A.3}$$

where $\hat{\psi}$ and $\hat{g}$ are unrenormalized quantities.

All divergencies can be absorbed by the bare coupling constant $\hat{g}$. The wave function renormalization constant Z_2 defined by

$$\langle T\,\psi_\alpha\bar{\psi}_\beta\rangle(p) \approx i\delta_{\alpha\beta}\,(\gamma p - m)^{-1} \quad \text{at } \gamma p \approx m \tag{A.4}$$

where m is the physical fermion mass, turns out to be finite (at least in the lowest nontrivial order). The Green's functions depend only on the parameter m and N (as we shall see below).

One rewrites eq. (A.2) introducing the auxiliary field $\sigma(x)$

$$Z(\xi,\bar{\xi}) = \int d\bar{\hat{\psi}}d\hat{\psi}d\sigma \exp i\left(\mathcal{A}\left(\bar{\hat{\psi}},\hat{\psi}\right) + \int d^2x\left(-\bar{\hat{\psi}}\sigma\hat{\psi} - \frac{\sigma^2}{2\hat{g}^2} + \bar{\xi}\hat{\psi} + \bar{\hat{\psi}}\xi\right)\right) \tag{A.5}$$

and perform the $\hat{\psi}$-integrations

$$Z(\xi,\bar{\xi}) = \int d\sigma \exp i\left(\mathcal{A}_{eff}\left(\sigma\right) - \bar{\xi}S(\sigma)\xi\right) \tag{A.6}$$

with the "effective action"

$$\mathcal{A}_{eff}\left(\sigma\right) = -iN\,\mathrm{Tr}\ln iS^{-1}(\sigma) - \int d^2x\frac{\sigma^2}{2\hat{g}^2} \tag{A.7}$$

and the fermion propagator in the field $\sigma(x)$

$$S_{\alpha\beta}(\sigma) = i\delta_{\alpha\beta}\left(i\gamma\partial - \sigma\right)^{-1}. \tag{A.8}$$

The Gross-Neveu model in $1/N$-expansion is defined by a perturbation expansion of eq. (A.5) around the (x-independent) stationary point $\sigma = m_0$ of the effective action (A.7)

$$\frac{\delta}{\delta\sigma(x)}\mathcal{A}_{eff}\left(\sigma\right) = N\,\mathrm{tr}\,S(x,x) - \frac{\sigma}{\hat{g}^2} = 0 \tag{A.9}$$

which implies

$$\sigma = m_0 = \Lambda e^{-\frac{\pi}{N\hat{g}^2}} \tag{A.10}$$

where Λ is an UV-cutoff parameter. The σ-propagator (in lowest oprder) is defined by

$$\begin{aligned} D^{-1}(x,y) &= -i\frac{\delta}{\delta\sigma(x)}\frac{\delta}{\delta\sigma(y)}\mathcal{A}_{eff}\left(\sigma\right)\Big|_{\sigma=m_0} \\ &= -N\,\mathrm{tr}\left(S_0(x,y)S_0(y,x)\right) + \frac{i}{\hat{g}^2}\delta(x-y) \end{aligned} \tag{A.11}$$

where the trace only refers to Dirac indices and

$$S_0 = S(\sigma = m_0) = i\,(i\gamma\partial - m_0)^{-1}\,. \tag{A.12}$$

This means in terms of Feynman diagrams that D^{-1} is a subtracted fermion loop, c.f. Fig 1 and A.9. In momentum space we obtain with eq. (A.10)Λ

$$\begin{aligned} D^{-1}(k^2, m_0) &= N \int \frac{d^2p}{(2\pi)^2}\,\mathrm{tr}\left[\frac{1}{\gamma p - m_0}\left(\frac{1}{\gamma p + \gamma k - m_0} - \frac{1}{m_0}\right)\right] \\ &= -\frac{N}{2\pi i}\phi \coth\frac{1}{2}\phi \end{aligned} \tag{A.13}$$

where $k^2 = -4m_0^2 \sinh^2 \frac{1}{2}\phi$.

$$\underset{\alpha \quad p \quad \beta}{\longleftarrow} = S_{0\alpha\beta} = \delta_{\alpha\beta}\frac{i}{\gamma p - m_0}\;,\quad \underset{k}{\sim\!\sim\!\sim} = D = -\frac{2\pi i}{N}\frac{\tanh\frac{1}{2}\phi}{\phi}\;,\quad \alpha \longleftarrow \beta = -i\delta_{\alpha\beta}$$

Figure 7: Feynman rules of the Gross-Neveu model

The (unrenormalized) fermion Green's function $\langle\, T\,\hat{\psi}(x_1)\ldots\bar{\hat{\psi}}(x_n)\,\rangle$ depending now only on the parameters m_0 and N are given by all Feynman diagrams which do not contain a loop with only one or more two fermion lines, constructed by fermion propagators $S_{\alpha\beta}(m_0)$, σ-propagators $D(m_0)$ and vertices $-i\delta_{AB}$, corresponding to the Yukawa interaction $-\bar{\hat{\psi}}\sigma\hat{\psi}$ in eq. (A.5), c.f Fig. 7. A diagram is of order N^{-n}if it contains k fermion loops and $n+k$ σ-propagators.

We give a short review of the renormalization of the Gross-Neveu model in $1/N$-expansion in lowest non-trivial order. In lowest order $m_0 = \langle\sigma\rangle$ id the fermion mass generated by spontaneous symmetry breaking. The exact value is given by

$$m = m_0 + \delta m \tag{A.14}$$

where δm is the fermion self-mass which is determined by the full fermion propagator

$$\begin{aligned} S'_{\alpha\beta}(p) &= \langle\, T\hat{\psi}_\alpha\bar{\hat{\psi}}_\beta\,\rangle(p) = i\delta_{\alpha\beta}\,(i\gamma\partial - m_0 - \Sigma(p))^{-1} \\ &= i\delta_{\alpha\beta}Z_2\,(i\gamma\partial - m_0 - \Sigma_c(p))^{-1} \end{aligned} \tag{A.15}$$

and the convergent self-energy $\Sigma_c(p) = O((\gamma p - m)^2)$. In order $1/N$ the function $\Sigma(p)$ is given by the diagrams of Fig. 8.

$$\begin{aligned} \Sigma^{(a)}(p) &= \int^\Lambda \frac{d^2k}{(2\pi)^2}\frac{D(k^2, m_0)}{\gamma p + \gamma k - m_0} \\ \Sigma^{(b)}(p) &= -N\int^\Lambda \frac{d^2k}{(2\pi)^2}\frac{d^2k'}{(2\pi)^2}\,\mathrm{tr}\,\frac{D(k^2, m_0)D(0, m_0)}{(\gamma k' - m_0)^2\,(\gamma k' + \gamma k - m_0)} \\ &= -\frac{2\pi i}{N}m_0\int^\Lambda \frac{d^2k}{(2\pi)^2}\frac{1}{k^2 - 4m_0^2}\,. \end{aligned} \tag{A.16}$$

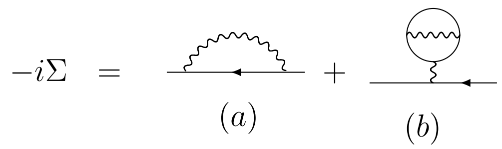

Figure 8: $O(N^{-2})$ fermion self-energy diagrams

Hence

$$\Sigma(p) = A(p^2)\gamma p - mB(p^2)$$

with

$$A(p^2) = \frac{1}{p^2}\int^\Lambda \frac{d^2k}{(2\pi)^2}\frac{D(k^2,m)\,(p^2+kp)}{(p+k)^2-m^2} + O(N^{-2}) \tag{A.17}$$
$$B(p^2) = \int^\Lambda \frac{d^2k}{(2\pi)^2}\left(\frac{D(k^2,m)}{(p+k)^2-m^2} - \frac{2\pi i}{N}\frac{1}{k^2-4m^2}\right) + O(N^{-2}).$$

The self-mass is given by

$$\frac{\delta m}{m} = A(m^2)+B(m^2) \approx -\frac{1}{n}\left(\ln\frac{\Lambda}{m}+\frac{1}{2}\ln\ln\frac{\Lambda}{m}\right) + O(N^{-2}) \tag{A.18}$$

and the fermion wave function renormalization constant Z_2 is finite [1]

$$\begin{aligned} Z_2^{-1}-1 &= -A(m^2) - 2m^2\left(A'(m^2)+B'(m^2)\right) \\ &= -\int \frac{d^2k}{(2\pi)^2} D(k^2,m)\,\frac{k^2-2(kp)^2/m^2-4m^2}{(k^2+2kp)^2}\bigg|_{p^2=m^2} + O(N^{-2})\,. \end{aligned} \tag{A.19}$$

In order to complete the renormalization in lowest non-trivial order, we have to discuss vertex and σ-propagator corrections. The full vertex function is

$$\begin{aligned} \Gamma_{AB} &= -i\delta_{AB} + \Lambda_{AB} \\ &= Z_1\left(-i\delta_{AB} + \Lambda_{c\ AB}\right) \end{aligned} \tag{A.20}$$

where Λ_{AB} in order $1/N$ is given by the graphs of Fig. 4.

The divergent contribution to Z_1 is

$$\begin{aligned} Z_1 - 1 &= \int^\Lambda \frac{d^2k}{(2\pi)^2}\frac{D(k^2,m)}{k^2-m^2} + O(N^{-2}) \\ &\approx -\frac{1}{2N}\ln\ln\frac{\Lambda}{m} + O(N^{-2})\,. \end{aligned} \tag{A.21}$$

The full σ-propagator

$$\begin{aligned} D'(k^2) &= \left(D^{-1}(k^2,m_0) - \Pi(k^2)\right)^{-1} \\ &= Z_3\left(D^{-1}(k^2,m) - \Pi_c(k^2)\right)^{-1} \end{aligned} \tag{A.22}$$

is analyzed in detail in section 3 and appendix B. Since (at least up to this order) $Z_1^2 Z_3$ is finite, the amputated four-point function

$$\langle\, T\hat{\psi}_A\bar{\hat{\psi}}_B\hat{\psi}_C\bar{\hat{\psi}}_D\,\rangle = \Gamma_{AB}D'\Gamma_{CD} + \text{ crossed term } + \text{ box diagram} \tag{A.23}$$

is convergent to order $1/N^2$. This means that there are no divergences in the fermion Green's functions after fixing the physical mass by eq. (A.4), as the wave function renormalization constant Z_2 is finite, too.

The nonlinear σ-model is defined by the Lagrangian

$$\mathcal{L}^{\mathrm{NLS}} = \frac{1}{2}\left(\partial_\mu n\right)^2 \text{ and the constraint } n^2 = 1/g\,. \tag{A.24}$$

The formulas analogous to (A.1) -(A.13) are

$$\langle\, T\, n_{i_1}(x_1)\ldots n_{i_n}(x_n)\,\rangle = Z_2^{-n/2} i^{-n} \frac{\delta}{\delta J_{i_1}(x_1)\ldots\delta J_{i_n}(x_n)} Z(J) \tag{A.25}$$

with the Feynman path integral

$$\begin{aligned} Z(J) &= \int d\hat{n}\,\delta(\hat{n}^2 - 1/\hat{g})\,\exp i\left(\frac{1}{2}\left(\partial_\mu n\right)^2 + J\hat{n}\right) \\ &= \int d\hat{n}\,d\omega\,\exp i\left(\frac{1}{2}\left(\partial_\mu n\right)^2 - \frac{1}{2}\omega(\hat{n}^2 - 1/\hat{g}) + J\hat{n}\right) \\ &= \int d\omega\,\exp\left(i\mathcal{A}_{eff}^{NLS}(\omega) - \frac{1}{2}J\Delta(\omega)J\right) \end{aligned} \tag{A.26}$$

where

$$\mathcal{A}_{eff}^{\mathrm{NLS}}(\omega) = \frac{i}{2}N\ \mathrm{Tr}\ln i\Delta^{-1}(\omega) + \int d^2x\frac{\omega}{2\hat{g}} \tag{A.27}$$

and

$$\Delta_{ij}(\omega) = i\delta_{ij}\left(-\Box - \omega\right)^{-1}.$$

The (x-independent) stationary point of the effective action is

$$\omega = \langle\omega\rangle = m_0^2 = \Lambda^2 e^{-\frac{4\pi}{N\hat{g}}} \tag{A.28}$$

and the ω-propagator (in lowest order)

$$D_\omega(k^2) = \frac{2}{N}\left(\Delta^2(m_0)\right)^{-1}(k^2) = \frac{8\pi i}{N} m_0^2 \frac{\sinh\phi}{\phi}\,. \tag{A.29}$$

The Feynman rules are listed in Fig. 9.

$$\underset{i \quad p \quad j}{- - - - - - -} = \Delta_{0ij} = \delta_{ij}\frac{i}{p^2 - m_0^2}\ , \quad \underset{k}{\sim\sim\sim} = D_\omega = \frac{8\pi i}{N} m_0^2 \frac{\sinh\frac{1}{2}\phi}{\phi}\ , \quad \underset{i \qquad j}{- - - \wr - - -} = -\tfrac{1}{2} i \delta_{ij}$$

Figure 9: Feynman rules of the nonlinear σ-model

B $1/N^2$-corrections to the σ-propagator

In this appendix we calculate $1/N^2$-corrections to the σ-propagator

$$D'(k^2) = \left(D^{-1}(k^2, m_0) - \Pi(k^2)\right)^{-1}. \tag{B.1}$$

The σ-self-energy $\Pi(k^2)$ in order N^0 is given by the Feynman diagrams of Fig. 2

$$\begin{aligned}
\Pi^{(a)}(k^2) &= -N \int \frac{d^2 l}{(2\pi)^2} D(l^2, m_0) \\
&\quad \times \int \frac{d^2 p}{(2\pi)^2} \operatorname{tr} \frac{1}{\gamma p - \gamma l - m_0} \frac{1}{\gamma p - \gamma l - \gamma k - m_0} \frac{1}{\gamma p - \gamma k - m_0} \frac{1}{\gamma p - m_0} \\
\Pi^{(b)}(k^2) &= -2N \int \frac{d^2 l}{(2\pi)^2} D(l^2, m_0) \\
&\quad \times \int \frac{d^2 p}{(2\pi)^2} \operatorname{tr} \frac{1}{\gamma p - \gamma l - m_0} \frac{1}{\gamma p - m_0} \frac{1}{\gamma p - \gamma k - m_0} \frac{1}{\gamma p - m_0} \\
\Pi^{(c)}(k^2) &= -2N \sum\nolimits^{(b)} \int \frac{d^2 p}{(2\pi)^2} \frac{1}{\gamma p - \gamma k - m_0} \frac{1}{(\gamma p - m_0)^2} \\
\Pi^{(d)}(k^2) &= -2N^2 \int \frac{d^2 l}{(2\pi)^2} D(l^2, m_0) D((k-l)^2, m_0) \\
&\quad \times \left(\int \frac{d^2 p}{(2\pi)^2} \operatorname{tr} \frac{1}{\gamma p - m_0} \frac{1}{\gamma p - \gamma l - m_0} \frac{1}{\gamma p - \gamma k - m_0} \right)^2 .
\end{aligned} \tag{B.2}$$

The fermion loop integrations can be done by means of the "cutting rule" [22] which reads for an arbitrary fermion loop with scalar coupling

$$\int \frac{d^2 p}{(2\pi)^2} \operatorname{tr} \prod_{i=1}^{n} \frac{1}{\gamma p - \gamma k_i - m_i} = \frac{-1}{2\pi i} \sum_{i<j} \frac{\theta_{ij}}{m_i m_j \sinh\theta_{ij}} \sum_{s=\pm 1} \frac{\operatorname{tr} \prod_{l=1}^{n} \left(\gamma p_{ij}^s - \gamma k_l + m_l\right)}{\prod_{l=i,j} \left(\left(p_{ij}^s - k_l\right)^2 - m_l^2\right)} \tag{B.3}$$

where θ_{ij} is defined by

$$(k_i - k_j)^2 = m_i^2 + m_j^2 - 2 m_i m_j \cosh\theta_{ij}$$

and

$$p_{ij}^{s\mu} = \frac{1}{2\,(k_i - k_j)^2} \left\{ (k_i - k_j)^2 (k_i + k_j)^\mu + \left(m_i^2 - m_j^2\right)(k_i - k_j)^\mu + s 2 m_i m_j \sinh\theta_{ij} \epsilon^{\mu\nu} (k_i - k_j)_\nu \right\}$$

The graph 2(a) contains a divergent term proportional to $D^{-1}(k^2, m)$, all the other divergences in $\Pi(k^2)$ are cancelled in eq. (B.1) by corresponding terms in $D^{-1}(k^2, m_0)$. After a great deal of algebra we obtain for the full σ-propagator (up to order N^{-2})

$$D'(k^2) = Z_3 \left(D^{-1}(k^2, m) - \Pi_c(k^2)\right)^{-1}$$

with the convergent σ-self-energy

$$\begin{aligned}\Pi_c(k^2) = \int \frac{d^2 l}{(2\pi)^2}\Bigg\{ & D^{-1}(k^2) D(l^2) \left(\frac{4l^4 k^2}{F(k,-l)F(k,l)} - \frac{2l^2 k^2}{F(k,-l)} + \frac{2}{l^2 - 4m^2}\right) \\ & + D^{-1}(k^2) D((k-l)^2) \frac{16m^2 (kl)^2 (k^2 - kl)(l^2 - kl)}{F(k,-l)} \\ & - \frac{2i}{\pi}\left(1 - \frac{\phi}{\sinh\phi}\right) D(l^2) \left(\frac{(k^2 - kl)(kl - l^2)}{F(k,-l)} + \left.\frac{4m^2 - k'l}{4m^2 (l^2 - k'l)}\right|_{k'^2 = 4m^2}\right) \\ & - 2k^2 \left(k^2 - 4m^2\right) l^2 \frac{3k^2 l^2 - 4(kl)^2 l^2 + k^4 l^2 + 12m^2 \left((kl)^2 - k^2 l^2\right)}{F(k,-l)F(k,l)(l^2 - 4m^2)} \\ & + \frac{D(l^2) D((k-l)^2)}{D^2(k^2) F^2(k,-l)} 8m^2 (kl)^2 \left(k^2 - kl\right)^2 + O(1/N)\Bigg\}\end{aligned} \tag{B.4}$$

where

$$F(k,l) = (k+l)^2 + 4m^2 \left((kl)^2 - k^2 l^2 + O(1/N)\right)$$

and $k^2 = -4m^2 \sinh^2 \frac{1}{2}\phi$. The σ-wave renormalization constant, which has its origin in graph 2(a), is

$$Z_3 \approx 1 - 2\int^{\Lambda} \frac{d^2 l}{(2\pi)^2} \frac{D(l^2)}{l^2 - 4m^2} \approx 1 + \frac{1}{N} \ln\ln\frac{\Lambda}{m} + O(1/N)\,. \tag{B.5}$$

At threshold only the first term in eq. (B.4), which also comes from graph 2(a) contributes

$$\Pi_c(4m^2) = \frac{i\pi}{2} + O(1/N)\,. \tag{B.6}$$

C T-matrix elements in order N^{-2}

In this appendix we give more detailed calculations concerning the T-matrix elements in order N^{-2}. First the box diagrams Fig. 9(a), 9(b) yield

$$\begin{aligned} T^{\mathrm{NLS}}_{\mathrm{Box}}(\theta, N) = & \frac{(-i)^4}{m^2} i^2 \int^{\Lambda} \frac{d^2 k}{(2\pi)^2} D^2_{\omega}(k_2) \left(\frac{1}{(p_1 - k)^2 - m^2} + \frac{1}{(p_1 + k)^2 - m^2}\right) \\ & \times \frac{1}{(p_2 + k)^2 - m^2} + O\left(\frac{1}{N^3}\right) \end{aligned} \tag{C.1}$$

$$\begin{aligned} T^{\mathrm{GN}}_{\mathrm{Box}}(\theta, N) = & \frac{(-i)^4}{m^2} i^2 \int^{\Lambda} \frac{d^2 k}{(2\pi)^2} D^2(k_2) \bar{u}(p_1) \left(\frac{1}{\gamma p_1 - \gamma k - m} + \frac{1}{\gamma p_1 + \gamma k - m}\right) u(p_1) \\ & \times \bar{u}(p_2) \frac{1}{\gamma p_2 + \gamma k - m} u(p_2) + O\left(\frac{1}{N^3}\right)\,. \end{aligned} \tag{C.2}$$

Retaining the cutoff Λ we first note that

$$
\begin{aligned}
T^{\mathrm{NLS}}_{\mathrm{Box}}(\theta,N) - 4T^{\mathrm{GN}}_{\mathrm{Box}}(\theta,N) = &-\frac{8}{m^2}\int^{\Lambda}\frac{d^2k}{(2\pi)^2}D^2(k_2,m)\\
&\times\frac{k^4\left(k^2-4m^2\right)^2-16\left(m^2k^2-(p_1k)^2\right)\left(m^2k^2-(p_2k)^2\right)}{\left(k^2-2p_1k\right)\left(k^2+2p_1k\right)\left(k^2-2p_2k\right)\left(k^2+2p_2k\right)}+O\left(\frac{1}{N^3}\right)\\
&= -\frac{8}{m^2}\int^{\Lambda}\frac{d^2k}{(2\pi)^2}D^2(k_2,m)\frac{k^2-2p_1k-8m^2}{k^2+2p_1k}+O\left(\frac{1}{N^3}\right).
\end{aligned}
$$

The later is an energy independent (albeit divergent) constant. To evaluate $T^{\mathrm{GN}}_{\mathrm{Box}}(\theta,N)$ in detail we first write a dispersion relation for $D^2(k_2,m)$

$$
D^2(k^2,m) = \frac{16}{N}\frac{m^2}{k^2-4m^2}+\frac{1}{2\pi i}\int_{4m^2}^{\infty}dM^2\frac{\operatorname{disc}D^2(M^2)}{M^2-k^2}\,. \tag{C.3}
$$

For $M^2 = 4m^2\cosh^2\frac{1}{2}\phi$ we have

$$
\operatorname{disc}D^2(M^2) = \frac{(2\pi i)^2}{N^2}\coth^2\frac{1}{2}\phi\left(\frac{1}{(i\pi-\phi)^2}-\frac{1}{(i\pi+\phi)^2}\right) \tag{C.4}
$$

and, hence, we obtain the representation

$$
D^2(k^2,m) = \frac{(2\pi)^2}{N^2}\int_{-\infty}^{\infty}d\phi\left(\frac{\phi}{\left(\phi^2+\pi^2\right)^2}\frac{\cosh^3\frac{1}{2}\phi}{\sinh\frac{1}{2}\phi}+\frac{1}{\pi^2}\delta(\phi)\right)\frac{4m^2}{k^2-4m^2\cosh^2\frac{1}{2}\phi}\,. \tag{C.5}
$$

Introducing (C.5) into (C.2) we thus obtain

$$
T^{\mathrm{GN}}_{\mathrm{Box}}(\theta,N) = -\frac{8\pi i}{N^2}\int_{-\infty}^{\infty}d\phi\left(\frac{\phi}{\left(\phi^2+\pi^2\right)^2}\frac{\cosh^3\frac{1}{2}\phi}{\sinh\frac{1}{2}\phi}+\frac{\delta(\phi)}{\pi^2}\right)\left(B(\theta,\phi)+B(i\pi-\theta,\phi)\right) \tag{C.6}
$$

where $B(\theta,\phi)$ is a one-loop integral and can be evaluated using the cutting rule (B.3)

$$
\begin{aligned}
B(\theta,\phi) &= \frac{2}{\pi i}\int d^2k\frac{\left(p_1k+2m^2\right)\left(p_2k+2m^2\right)}{\left(k^2-4m^2\cosh\frac{1}{2}\phi\right)\left(k^2+2p_1k\right)\left(k^2+2p_2k\right)}\\
&= \frac{\cosh^2\frac{1}{2}\phi\tanh^4\frac{1}{2}\phi}{\cosh^2\frac{1}{2}\theta-\cosh^2\frac{1}{2}\phi}\left(\cosh^2\frac{1}{2}\theta\frac{\theta}{\sinh\theta}-\cosh^2\frac{1}{2}\phi\frac{\phi}{\sinh\phi}\right)\\
&+\frac{\theta}{\sinh\theta\cosh^4\frac{1}{2}\phi}\left(2\cosh^2\frac{1}{2}\phi-\cosh^2\frac{1}{2}\theta\right)\\
&+\ln\left(2\cosh\frac{1}{2}\phi\right)-\frac{\phi\sinh\frac{1}{2}\phi}{2\cosh^3\frac{1}{2}\phi}\left(\cosh^2\frac{1}{2}\phi+1\right)
\end{aligned} \tag{C.7}
$$

We insert this expression into (C.6) and use for the rational parts of the integrand formulas of the type

$$
\frac{\phi^2}{\left(\phi^2+\pi^2\right)^2} = \frac{1}{2\pi}\int_0^{\infty}dx\left(1-\pi x\right)e^{-\pi x}\cos\phi x\,.
$$

By interchanging the x- and ϕ-integrations we obtain the desired result (4.3).

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