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Generalized Jost functions and Levinson's theorem in a (1 + 1)-dimensional relativistic field-theoretic model

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By introducing generalized Jost functions with simple analytic properties we express the exact S matrix of the massive Thirring model (sine-Gordon model) for the fermion-fermion and the two fermion-antifermion channels (soliton-soliton and soliton-antisoliton) in a form very similar to that encountered in one-dimensional potential scattering. Relations between the number of bound states and the phase shifts of the direct and crossed channels (generalized Levinson's theorem) are discussed within this relativistic model.

I. INTRODUCTION AND RESULTS

At the semiclassical level there has been a previous investigation of the phase shifts for the sine-Gordon theory and comparisons were made with those of reflectionless potential scattering by Jackiw and Woo[1] and by Coleman.[2] Since then the exact S matrix has been found for the massive Thirring model[3] which is equivalent to the sine-Gordon model.[4]

In this paper we discuss the relation between the number of bound states and the scattering phase shifts (generalized Levinson's theorem) in the fermion-fermion and fermion-antifermion sectors on the basis of the exact S matrix and compare it with analogous results obtained in the case of potential scattering including reflection.

There already exists a discussion of one-space-dimensional potential scattering in the literature.[5] In Sec. II we give a different version by concentrating on the eigenvalues $s_\pm$ of the S matrix and by introducing Jost functions[6] for these eigenvalues. (For a symmetric potential, the two eigenvalues $s_\pm$ correspond to eigenstates of different parity.) This enables us to derive Levinson's theorem[7] in one-dimensional potential scattering for the even- and odd-parity phase shifts $\delta_\pm(k)$ and the numbers $n_\pm$ of bound states

$$n_\pm = \frac{1}{\pi}[\delta_\pm(0) - \delta_\pm(\infty)] \mp \tfrac{1}{4}[s_\pm(0) - 1], \quad s_\pm = e^{2i\delta_\pm(k)}, \quad (1)$$

where the allowed values for the pair $(s_+(0), s_-(0))$ are $(-1, 1)$, $(1, 1)$, and $(-1, -1)$. This formula is completely analogous to the original Levinson's theorem in three dimensions. There is, however, an important difference in the analytic properties of the S matrix. In three dimensions the eigenvalues of the S matrix may have so-called redundant poles in the physical sheet which do not correspond to bound states.[6] This is also true for the functions $s_\pm(k)$ in one dimension but not for the transmission amplitude $t(k) = \frac{1}{2}[s_+(k) + s_-(k)]$. The set of all poles in the physical sheet of the transmission amplitude corresponds to the set of all (even- and odd-parity) bound states. The energy values of all bound states are, therefore, fixed by the scattering amplitudes in contrast to the three-dimensional case.

Our main concern in this paper is the relativistic massive Thirring model [with coupling constant $g = (\pi/2)(\lambda - 1)$ and mass m] which is reviewed in Sec. III A. For the known exact (on-shell) solution[3] we show in Sec. III B that one can introduce generalized Jost functions such that the discussion of analytic properties (zeros, redundant poles, bound states) is nearly the same as in potential scattering. In particular the remarkable property of the transmission amplitude having poles only at the positions of the odd and even bound states is also true in the relativistic model. The essential difference is the crossing relation between the fermion-fermion channel (phase shift δ and no bound states) and the two fermion-antifermion channels which are even and odd under charge conjugation and intrinsic parity (phase shifts $\delta_\pm$ and numbers of bound states $n_\pm$). This changes Levinson's theorem to

$$n_\pm = \frac{1}{\pi}[\delta_\pm(\text{thr}) - \delta_\pm(\infty)] + \frac{1}{\pi}[\delta(\text{thr}) - \delta(\infty)] \pm \frac{1}{4}[s_\pm(\text{thr}) - 1], \quad (2)$$

$$s_\pm = e^{2i\delta_\pm}$$

where $\delta(\text{thr})$ is the phase shift at threshold and $(s_+(\text{thr}), s_-(\text{thr})) = (1, -1)$, $(1, 1)$, and $(-1, -1)$ for λ not an integer, λ an odd integer, and λ an even integer, respectively. For the case of integer values of λ there is no reflection and we have $\delta_+ \equiv \delta_- (\text{mod}\,\pi)$ and $\delta_+ - \delta_+(\text{thr}) \equiv \delta (\text{mod}\,\pi)$. This means that there is only one phase shift for all three channels.

The numbers of bound states in the model depend on λ as follows: $n_+ = [\lambda/2]$, $n_- = [(\lambda + 1)/2]$, $n_+ + n_-$

$=[\lambda]$, where the symbol $[\alpha]$ denotes the largest integer smaller than α.

The dependence of the phases $\delta_\pm$ and δ on the parameters λ and m are given explicitly in Sec. III B. Their asymptotic values are not the free field values but depend on λ in contrast to potential scattering. This means that even for arbitrarily high energies there is a nonzero scattering.

Note that the role of s_+ and s_- for the fermion case in Eq. (2) is interchanged as compared to the boson case in Eq. (1). This is due to the fact that a fermion-antifermion state with both particles at rest has odd parity.

In summary, the two-particle S matrix of the massive Thirring model has a simple structure and far-reaching similarities to nonrelativistic potential scattering in spite of its relativistic kinematics.

II. POTENTIAL SCATTERING

In this section we consider potential scattering in one space dimension[5] in close analogy to the well-known theory in three dimensions.[7] We confine ourselves to suitably well-behaved symmetric potentials, or, more precisely, potentials satisfying

$$\int_{-\infty}^{\infty} dx\,(1+|x|)\,|V(x)|<\infty \text{ and } V(x)=V(-x).$$

We consider the Jost solution $f(x,k)$ of the Schrödinger equation

$$-\frac{1}{2m}\,\psi_{xx}+V(x)\psi=E\psi,\quad E=\frac{k^2}{2m}$$

with the boundary condition

$$f(x,k)\approx e^{ikx} \text{ as } x\to+\infty. \tag{3}$$

It satisfies

$$f^*(x,k)=f(x,-k^*) \tag{4}$$

and is analytic in the upper half of the complex k plane with the estimate

$$f(x,k)=e^{ikx}+o(e^{-x\,\mathrm{Im}k}) \tag{5}$$

for $|k|\to\infty$. This can be seen by iterating the Volterra integral equation of the Jost solution. Since for real values of k the functions $f(x,k)$ and $f(x,-k)$ are linearly independent, they form a fundamental system. The even- and odd-parity solutions of the Schrödinger equation and the solution corresponding to an incoming wave only from the right are given by

$$\psi_\pm(x,k)=f(x,-k)\pm s_\pm f(x,k)=\pm f(-x,-k)+s_\pm f(-x,k),$$

$$\psi_r(x,k)=f(x,-k)+rf(x,k)=tf(-x,k),$$

where t and r are the transmission and reflection coefficients and $s_\pm=t\pm r$ are the eigenvalues of the S matrix obeying the unitarity relation $|s_\pm|=1$ for $E\geq0$ or real k. They may be expressed in terms of Wronskians evaluated at $x=0$ or $x=\infty$,

$$s_+(k)=\frac{W(f(x,k),f(x,-k)-f(-x,-k))}{W(f(x,k),f(-x,k))}=-\frac{f_x(0,-k)}{f_x(0,k)}, \tag{6a}$$

$$s_-(k)=\frac{W(f(x,k),f(x,-k)+f(-x,-k))}{W(f(x,k),f(-x,k))}=\frac{f(0,-k)}{f(0,k)}, \tag{6b}$$

$$t(k)=\frac{W(f(x,k),f(x,-k))}{W(f(x,k),f(-x,k))}=\frac{2ik}{2f_x(0,k)f(0,k)}. \tag{6c}$$

Let us introduce the Jost functions

$$f_+(k)=\frac{f_x(0,k)}{ik}, \tag{7a}$$

$$f_-(k)=f(0,k), \tag{7b}$$

leading to the relations

$$s_+(k)=\frac{f_+(-k)}{f_+(k)}, \tag{8a}$$

$$s_-(k)=\frac{f_-(-k)}{f_-(k)}, \tag{8b}$$

$$t(k)=\frac{1}{f_+(k)f_-(k)}. \tag{8c}$$

The properties

$$f_\pm^*(k)=f_\pm(-k^*) \tag{9a}$$

and

$$\lim_{|k|\to\infty} f_\pm(k)=1 \text{ for } \mathrm{Im}\,k\geq0 \tag{9b}$$

follow from Eq. (4) and (5). For real positive k we write

$$s_\pm(k)=e^{2i\delta(k)}. \tag{10}$$

defining the real phase shifts $\delta_\pm(k)$ modulo π from the observable S matrix. If the phase shifts are defined [consistently with Eq. (10)] by

$$f_\pm(k)=|f_\pm(k)|\,e^{-i\delta_\pm(k)}, \tag{11}$$

they are known modulo 2π for given $f_\pm(k)$.

It can be shown[6] that in the upper half of the k plane the functions $f_\pm(k)$ can have only a finite number of simple zeros on the imaginary axis which are different for f_+ and f_-. If $f_+(i\kappa)=0$ or $f_-(i\kappa)=0$, the Wronskian $W(f(x,i\kappa),\ f(-x,i\kappa))=2ik\,f_+(i\kappa)\,f_-(i\kappa)$ vanishes, which tells us that $f(x,i\kappa)$ is proportional to $f(-x,i\kappa)$ with the asymptotic behavior

$$f(x,i\kappa)\approx\begin{cases}e^{-\kappa x} & \text{as } x\to+\infty\\ \pm e^{\kappa x} & \text{as } x\to-\infty,\end{cases} \tag{12}$$

Hence, the zeros of the Jost function $f_\pm(k)$ at $k = i\kappa$, $\kappa > 0$, correspond to bound states with even or odd wave functions, respectively. In the upper half of the k plane the transmission coefficient $t(k)$ is nonzero and has only poles corresponding precisely to the even and odd bound states. The spectrum of all bound states can, therefore, be extracted from the transmission coefficient $t=\frac{1}{2}(s_+ + s_-)$ in contrast to the three-dimensional case.[6] The reflection coefficient r and the functions $s_\pm$, however, may have both zeros and redundant poles originating from the functions $f_\pm(-k)$ in the numerator of Eqs. (8a) and (8b).

We now derive Levinson's theorem which relates the scattering phase shifts $\delta_\pm$ to the numbers $n_\pm$ of bound states with even and odd parity, respectively. As the functions $f_\pm(k)$ are regular in the upper half of the plane, we can apply the argument principle with a contour C enclosing the upper half of the plane to get

$$\begin{aligned} n_\pm &= \frac{1}{2\pi i}\int_C dk\, \frac{f_\pm'(k)}{f_\pm(k)} \\ &= \frac{1}{2\pi i}\int_0^\infty dk\left[\ln\frac{f_\pm(k)}{f_\pm(-k)}\right]' - \tfrac{1}{2}\alpha_\pm \\ &= \frac{1}{\pi}\left[\delta_\pm(0) - \delta_\pm(\infty)\right] - \tfrac{1}{2}\alpha_\pm . \end{aligned} \tag{13}$$

Here the properties (9b) and (10) have been used. The term $-\frac{1}{2}\alpha_\pm$ arises from a small detour around $k=0$ where $f_\pm(k)$ has the behavior $f_\pm(k) \propto k^{\alpha_\pm}$ for $k\to 0$ or $s_\pm(0) = (-1)^{\alpha_\pm}$. Since $kf_+(k)$ and $f_-(k)$ are continuous[5] and $|f_+f_-| = 1/|t| \geq 1$, we have the three conditions $\alpha_+ \geq -1$, $\alpha_- \geq 0$, and $\alpha_+\alpha_- \leq 0$, which [if the functions $f_\pm(k)$ are meromorphic at the origin] allow only the pairs

$$(\alpha_+, \alpha_-) = (-1, 0),\ (-1, 1),\ \text{and } (0, 0) \tag{14}$$

corresponding to the threshold values

$$(s_+(0), s_-(0)) = (-1, 1),\ (-1, -1),\ \text{and } (1, 1), \tag{15}$$

respectively. The correspondence between (14) and (15) may be expressed by the relation

$$\alpha_\pm = \pm\tfrac{1}{2}[s_\pm(0) - 1]. \tag{16}$$

Equations (13) and (16) give Levinson's theorem in one-dimensional potential scattering in the form of Eq. (1).

The total number of bound states $n_+ + n_-$ could have been obtained directly by applying the argument principle to the transmission coefficient $t(k)$ [cf. (8c)] which is meromorphic and nonzero in the upper half plane. This shows that the number $n_+ + n_-$ is determined by the phase $\delta_+(k) + \delta_-(k)$ of $t(k)$ and the threshold behavior $t(k) \propto k^{-\alpha_+ - \alpha_-}$ for $k \to 0$ which could also be checked from Eq. (13). This is peculiar to scattering in one space dimension.

An interesting case is the reflectionless potential with $r(k) = \frac{1}{2}[s_+(k) - s_-(k)]$ identical to zero. Then we have

$$s_+(k) = s_-(k) = t(k), \tag{17}$$

in spite of the different bound states in the two channels. From Eqs. (17) and (8) we get

$$f_-(k) = [f_+(-k)]^{-1}, \tag{18}$$

which implies that $s_+(k)$ has redundant poles [poles of $f_+(-k)$] at the odd bound-state energies [zeros of $f_-(k)$] and vice versa. An example for a reflectionless case is the potential

$$V(x) = \frac{\alpha^2}{2m}\,\frac{\lambda(\lambda-1)}{\cosh^2\alpha x}$$

for λ a positive integer.[8]

III. SINE-GORDON THEORY OR MASSIVE THIRRING MODEL

In this section we generalize the discussion of Sec. II to a relativistic field model in 1+1 dimensions, the sine-Gordon theory, which is characterized by the equation

$$\Box\phi = -\frac{\alpha}{\beta}\sin\beta\phi . \tag{19}$$

It has been established[4] that this theory is equivalent to the massive Thirring model defined by

$$i\gamma_\mu\partial^\mu\psi - m\psi = g\,\gamma_\mu\psi\bar{\psi}\gamma^\mu\psi . \tag{20}$$

In the following we use, instead of g, the coupling parameter λ related to g by

$$\lambda = 1 + \frac{2g}{\pi}. \tag{21}$$

In particular the soliton, antisoliton, and the elementary boson of the sine-Gordon theory may be identified with the elementary fermion, antifermion, and the lowest fermion-antifermion bound state [cf. Eq. (34)] of the massive Thirring model, and the coupling constants are related by

$$\lambda = \frac{8\pi}{\beta^2} - 1 . \tag{22}$$

In this section we shall use the language of the Thirring model and restrict ourselves to the scattering of two particles (fermion, antifermion). As there is no particle production in this model we have only elastic scattering.

A. The Two-Particle Scattering Functions of the Model

The scattering of a fermion and an antifermion

$$f(p_1) + \bar{f}(p_2) \to f(p_1') + \bar{f}(p_2')$$

is given by boundary values of two analytic functions $t(\theta)$ and $r(\theta)$, the forward- and the backward-scattering amplitudes, respectively,

$$^{out}\langle f(p_1')\bar{f}(p_2')|f(p_1)\bar{f}(p_2)\rangle^{in} = \delta(p_1^{1'}-p_1^1)\delta(p_2^{1'}-p_2^1)t(\rho) - \delta(p_2^{1'}-p_1^1)\delta(p_1^{1'}-p_2^1)r(\rho)\,, \quad (23)$$

where ρ is the modulus of the rapidity difference, $\rho=|\rho_1-\rho_2|$, of the incoming fermion and antifermion ($p_i^1=m\sinh\rho_i$) which is connected to the energy by

$$(p_1+p_2)^2=2m^2(1+\cosh\rho)\,. \quad (24)$$

Let us consider the conformal mapping $s=2m^2(1+\cosh\theta)$ which relates the Mandelstam variable s in the cut physical plane to the complex rapidity difference θ in the strip $0<\text{Im}\theta<\pi$. It maps the upper (lower) rim of the positive $(s-4m^2)$ axis onto the positive (negative) θ axis and the lower (upper) rim of the negative s axis onto the negative (positive) $(\theta-i\pi)$ axis. The prescription $\theta=\rho=|\rho_1-\rho_2|$, therefore, corresponds to the usual $s+i\epsilon$ rule. The functions $t(\theta)$ and $r(\theta)$ of the massive Thirring model have the remarkable property of being meromorphic in the whole θ plane.

We can diagonalize the S matrix (23) by introducing eigenstates of the charge conjugation operator and obtain the corresponding eigenvalues

$$s_\pm(\theta)=t(\theta)\pm r(\theta)\,. \quad (25)$$

For $\theta=\rho\geq 0$, the unitarity of the S matrix gives real phase shifts $\delta_\pm(\rho)$ defined by

$$s_\pm(\rho)=e^{2i\delta_\pm(\rho)}, \quad \rho\geq 0\,. \quad (26)$$

For the scattering of identical fermions (antifermions)

$$f(p_1)+f(p_2)\to f(p_1')+f(p_2')\,,$$

there is no distinction between forward and backward scattering. By means of crossing symmetry this process is described by another boundary value of the function $t(\theta)$

$$^{out}\langle f(p_1')f(p_2')|f(p_1)f(p_2)\rangle^{in} = [\,\delta(p_1^{1'}-p_1^1)\delta(p_2^{1'}-p_2^1) - \delta(p_2^{1'}-p_1^1)\delta(p_1^{1'}-p_2^1)]\,t(i\pi-\sigma)\,, \quad (27)$$

where σ is the modulus of the rapidity difference $\sigma=|\sigma_1-\sigma_2|$ of the incoming fermions ($p_i^1=m\sinh\sigma_i$). Unitarity leads to

$$t(i\pi-\sigma)=e^{2i\delta(\sigma)}, \quad \sigma\geq 0 \quad (28)$$

with the real phase shift $\delta(\sigma)$ for fermion-fermion scattering. Different boundary values of the function $t(\theta)$ describe different processes and for the function $r(\theta)$ crossing symmetry leads to the condition

$$r(\theta)=r(i\pi-\theta)\,. \quad (29)$$

Both functions t and r are real analytic in the variable $i\theta$,

$$t^*(\theta)=t(-\theta^*), \quad r^*(\theta)=r(-\theta^*)\,. \quad (30)$$

B. Generalized Jost Functions

The explicit form of the amplitudes $t(\theta)$ and $r(\theta)$ were first given by Zamolodchikov[3]

$$t(i\pi-\theta)=\frac{g(-\theta)}{g(\theta)}\,, \quad t(\theta)=\frac{\sinh\lambda\theta}{\sinh\lambda(i\pi-\theta)}\,t(i\pi-\theta)\,,$$

$$r(\theta)=\frac{\sinh\lambda i\pi}{\sinh\lambda\theta}\,t(\theta)\,, \quad (31)$$

where

$$g(\theta)=\prod_{l=0}^{\infty}\prod_{k=1}^{\infty}\frac{(2l+1+k/\lambda+\theta/i\pi)[2l+1+(k-1)/\lambda+\theta/i\pi]}{(2l+k/\lambda+\theta/i\pi)[2l+2+(k-1)/\lambda+\theta/i\pi]}\,. \quad (32)$$

Another form is due to Weisz[9]

$$t(i\pi-\theta)=\exp\left(-\int_0^\infty\frac{dx}{x}\,\frac{\sinh(x/2)(1-\lambda)}{\sinh(x/2)\cosh(x/2)\lambda}\times\sinh x\lambda\frac{\theta}{i\pi}\right)\,, \quad (33)$$

valid for $|\text{Im}\theta|<\pi\min(1/\lambda,1)$. By inspection of Eqs. (31) and (32) one sees that in the physical strip the function $t(\theta)$ has no zeros and only simple poles at the positions

$$\theta_k=i\pi(1-k/\lambda), \quad k=1,2,\ldots<\lambda \quad (34)$$

corresponding to the energies

$$m_k=2m\sin\frac{\pi k}{2\lambda}\,.$$

These values coincide with the spectrum which was calculated in the WKB approximation[10] if the parameter λ is related to the coupling constants β and g as in Eqs. (21) and (22). The energies m_k are known as the masses of the bound states in the even and odd channels for even and odd values of k, respectively. The transmission amplitude $t(\theta)$ has only these poles which is already reminiscent of potential scattering.

The amplitudes $s_\pm(\theta)$ [Eqs. (25) and (31)] are explicitly given by

$$s_+(\theta)=-\frac{\sinh(\lambda/2)(\theta+i\pi)}{\sinh(\lambda/2)(\theta-i\pi)}\,t(i\pi-\theta)\,,$$
$$s_-(\theta)=-\frac{\cosh(\lambda/2)(\theta+i\pi)}{\cosh(\lambda/2)(\theta-i\pi)}\,t(i\pi-\theta)\,. \quad (35)$$

With Eq. (32) and the product expansion of the hyperbolic functions one can write simultaneously

$$s_\pm(\theta)=\frac{g_\pm(-\theta)}{g_\pm(\theta)} \tag{36}$$

and

$$t(\theta)=\frac{1}{g_+(\theta)g_-(\theta)} \ . \tag{37}$$

The functions $g_\pm(\theta)$, obtained by splitting the products over k [cf. Eq. (32)] into even and odd k, are regular in the physical strip and have zeros precisely at the positions of the bound states of the corresponding channel:

$$g_+(\theta)=\prod_{l=0}^{\infty}\prod_{k=1}^{\infty}\frac{(2l-1+2k/\lambda+\theta/i\pi)[2l+1+(2k-1)/\lambda+\theta/i\pi]}{(2l+2k/\lambda+\theta/i\pi)[2l+(2k-1)/\lambda+\theta/i\pi]} \times\frac{[2l+2+(2k+1)/\lambda-\theta/i\pi][2l+2+(2k-2)/\lambda-\theta/i\pi]}{[2l+1+(2k+1)/\lambda-\theta/i\pi][2l+3+(2k-2)/\lambda-\theta/i\pi]}\ , \tag{38}$$

$$g_-(\theta)=g_+(\theta-i\pi/\lambda) \text{ and } g_-(i\pi-\theta)=[g_+(i\pi+\theta)]^{-1}\ . \tag{39}$$

An integral representation similar to Eq. (33) reads

$$g_\pm(i\pi-\theta)=\exp\left[\int_0^\infty\frac{dx}{x}\,\frac{\sinh(x/2)(1-\lambda)}{\sinh x\cosh(x/2)\lambda}\times\sinh x\left(\frac{\lambda\theta}{i\pi}\mp\frac{1}{2}\right)\right], \tag{40}$$

valid for $|(\lambda/\pi)\mathrm{Im}\theta\mp\frac{1}{2}|<\frac{1}{2}+\min(\lambda,1)$.

In view of Eqs. (36) and (37) we would like to call the functions $g_\pm(\theta)$ generalized Jost functions. They are real analytic in the variable $i\theta$ and their asymptotic values are given by Eq. (40):

$$\lim_{\mathrm{Re}\,\theta\to\infty} g_\pm(\theta)=\exp[-i\tfrac{\pi}{4}(\lambda-1)] \text{ independent of } \mathrm{Im}\theta. \tag{41}$$

In view of Eqs. (25), (28), (30), and (37), this leads to

$$\delta_+(\infty)\equiv\delta_-(\infty)\equiv-\delta(\infty)\equiv\frac{\pi}{4}(\lambda-1)\quad(\mathrm{mod}\,\pi)\ . \tag{42}$$

The relations (39) between g_+ and g_- may be combined with Eq. (37) to give

$$t(i\pi-\theta)=\frac{g_+(i\pi+\theta)}{g_+(i\pi-\theta)}=\frac{g_-(i\pi+\theta)}{g_-(i\pi-\theta)}\ . \tag{43}$$

Comparing Eqs. (36) and (37) for the fermion-antifermion channels with Eqs. (8a), (8b), and (8c), and comparing (39) and (43) for the fermion-fermion channel with Eqs. (17) and (18), we observe a far-reaching analogy between the relativistic model and potential scattering.

C. Generalized Levinson's Theorem

The numbers $n_\pm$ of bound states in the even and odd channels are equal to the number of zeros of the generalized Jost functions $g_\pm(\theta)$ in the physical strip. Since the zeros are simple, we get, by using the argument principle for a contour C enclosing the physical strip,

$$\begin{aligned} n_\pm &=\frac{1}{2\pi i}\int_C d\theta\,\frac{g'_\pm(\theta)}{g_\pm(\theta)} \\ &=\frac{1}{2\pi i}\int_0^\infty d\rho\left(\ln\frac{g_\pm(\rho)}{g_\pm(-\rho)}\right)' +\frac{1}{2\pi i}\int_0^\infty d\sigma\left(\ln\frac{g_\pm(i\pi+\sigma)}{g_\pm(i\pi-\sigma)}\right)'-\tfrac{1}{2}\alpha_\pm \\ &=\frac{1}{\pi}[\delta_\pm(0)-\delta_\pm(\infty)]+\frac{1}{\pi}[\delta(0)-\delta(\infty)]-\tfrac{1}{2}\alpha_\pm, \end{aligned} \tag{44}$$

using Eqs. (26), (28), (41) and (43). The terms $-\frac{1}{2}\alpha_\pm$ result from a detour around $\theta=0$, where $g_\pm(\theta)$ may have zeros or poles. At $\theta=i\pi$ the functions $g_\pm(\theta)$ are regular and different from zero. From the representation (33) one sees that

$$g_\pm(\theta)\propto\theta^{\alpha_\pm} \text{ for } \theta\to0\ ,$$

with

$$(\alpha_+,\alpha_-)=(0,-1),\ (1,-1),\ \text{and } (0,0), \tag{45}$$

corresponding to

$$(s_+(0),s_-(0))=(1,-1),\ (-1,-1),\ \text{and } (1,1),$$

if λ is a noninteger, an even, and an odd integer, respectively. The correspondence can be expressed by the relation

$$\alpha_\pm=\mp\tfrac{1}{2}[s_\pm(0)-1] \tag{46}$$

leading to the generalized Levinson's theorem stated in Eq. (2).

The total number $n_+ + n_-$ given by Eq. (44) depends only on $t(\theta)$ since $\delta_+(\rho)+\delta_-(\rho)$ is the phase of $t(\rho)$, $(\rho\geq0)$, $\delta(\sigma)$ is the phase of $t(i\pi-\sigma)$ $(\sigma\geq0)$, and $t(\theta)\propto\theta^{-\alpha_+-\alpha_-}$ for $\theta\to0$, due to Eq. (37). It is notable that we have exactly the same peculiar situation as in one-dimensional potential scattering, namely there are no zeros and redundant poles in the forward-scattering amplitude.

Within our model we can discuss the contributions to Eq. (44) from the fermion-antifermion and fermion-fermion channels separately. From Eqs. (33) and (35) we get for noninteger values of λ the following:

$$\frac{1}{\pi}(\delta_+(0)-\delta_+(\infty))=\left[\frac{\lambda}{2}\right]-\tfrac{1}{4}(\lambda-1),$$

$$\frac{1}{\pi}(\delta_-(0)-\delta_-(\infty))=\left[\frac{\lambda+1}{2}\right]-\tfrac{1}{2}-\tfrac{1}{4}(\lambda-1), \quad (47)$$

$$\frac{1}{\pi}(\delta(0)-\delta(\infty))=\tfrac{1}{4}(\lambda-1),$$

where $[\alpha]$ is the largest integer smaller than α, and for integer λ

$$\frac{1}{\pi}(\delta_\pm(0)-\delta_\pm(\infty))=\frac{1}{\pi}(\delta(0)-\delta(\infty))=\frac{1}{4}(\lambda-1). \quad (48)$$

With Eqs. (44) and (45) the explicit values (47) give, of course, the known numbers [determined by Eq. (34)]:

$$n_+=\left[\frac{\lambda}{2}\right], \quad n_-=\left[\frac{\lambda+1}{2}\right], \quad n_++n_-=[\lambda].$$

In the semiclassical limit ($\lambda\to\infty$), we have

$$\frac{1}{\pi}(\delta_+(0)-\delta_+(\infty))=\frac{1}{\pi}(\delta_-(0)-\delta_-(\infty))=\frac{1}{\pi}(\delta(0)-\delta(\infty))=\frac{\lambda}{4}+O(1),$$

in accordance with Ref. 1, Eqs. (3), (8a), and (8b).

The ambiguity of phases (modπ) cancels in the differences in Eqs. (44) and (47) since $\delta(0)$ and $\delta(\infty)$ are defined as boundary values a continuous function $\delta(\sigma)$ for $\sigma\geq 0$. It may be convenient to remove the ambiguities in the phases as in Ref. 1. For the free theory ($\lambda=1$) the S matrix is equal to one. A natural convention is, therefore, to consider $\delta(\sigma,\lambda)$ and $\delta\pm(\rho,\lambda)$, ($\rho>0$), as continuous functions of λ which vanish identically for $\lambda=1$. This prescription gives

$$\delta(\sigma)=-\tfrac{1}{2}\int_0^\infty\frac{dx}{x}\,\frac{\sinh x(\lambda-1)}{\sinh x\cosh x\lambda}\sin 2x\frac{\sigma}{\pi} \quad (49)$$

and, in view of Eq. (35),

$$\delta_\pm(\rho)=\delta(\rho)+\frac{\pi}{2}(\lambda-1)+\sum_{n=1}^{\infty}(\pm 1)^n\,\frac{1}{n}\,e^{-n\lambda\rho}\sin\lambda\pi n. \quad (50)$$

The well-known semiclassical limit of the phase shifts (see, e.g., Ref. 1) is the first term in the expansion

$$\delta(\sigma)=-\frac{\lambda}{2}\int_0^\infty\frac{dx}{x}\tanh x\sin x\frac{2\sigma}{\pi}+\frac{\pi}{4}+O\left(\frac{1}{\lambda}\right)\text{ for }\sigma\neq 0.$$

This shows that even in the limit $\lambda\to\infty$ the scattering amplitude does not have the semiclassical value, the term $\pi/4$ survives for $\sigma\neq 0$.

Since

$$\delta_+(\infty)=\delta_-(\infty)=\delta(\infty)=\frac{\pi}{4}(\lambda-1),$$

we get from Eqs. (44) and (50)

$$\frac{1}{\pi}\delta_\pm(0)=n_\pm+\tfrac{1}{2}\alpha_\pm\text{ and }\delta(0)=0. \quad (51)$$

The latter equality may be related to the absence of fermion-fermion bound states. Equations (51) are identical to Eqs. (13) if the phase shifts in potential scattering are defined relative to the free case which, in particular, means $\delta(\infty)=0$. Note that with this convention Levinson's theorem (51) relates the numbers of bound states to the phases at threshold of the corresponding channels only, in contrast to the general formulation (44) where both the direct and the crossed channel contribute. This is in agreement with the semiclassical Levinson's theorem of Ref. 1.

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