ON THE UNIQUENESS OF A PURELY ELASTIC S-MATRIX IN (1+1) DIMENSIONS

M. KAROWSKI, H.J. THUN, T.T. TRUONG and P.H. WEISZ
Institut für Theoretische Physik, Freie Universität Berlin, Germany

Received 23 February 1977

Absence of particle production, non-vanishing backward scattering and factorization are shown to determine uniquely the S-matrix.

In this letter we show that the S-matrix of a (1+1) dimensional model of a single massive fermion f is uniquely determined if we assume absence of particle production, non-vanishing backward particle-antiparticle scattering and factorization [1] into 2-body S-matrices. The absence of particle production and factorization [2] are known to be implied by the existence of an infinite set of conserved local currents [3]. The S-matrix thus obtained is that of the massive Thirring model (sine-Gordon theory) proposed recently by Zamolodchikov [4]. However, this author derives his result on the basis of the following assumptions:

(a) Meromorphy of all 2-body scattering amplitudes in the complex plane of the rapidity difference of the two momenta.

(b) Exactness of the quasi-classical bound state spectrum.

(c) Vanishing of the 2-body reflection amplitude at integer values of the "coupling" constant λ.

(d) Absence of resonances.

Here in our treatment only assumption (a) is adopted; it will be seen that (b), (c), and (d) are consequences of the general properties.

For convenience, we shall consider the particle-antiparticle transmission and reflection amplitudes as functions of the rapidity difference $\theta_1-\theta_2=\theta_{12}$ (= θ for simplicity) of the momenta p_1 and p_2. These amplitudes $t_{f\bar{f}}(\theta)$ and $r_{f\bar{f}}(\theta)$ are defined by

$$^{\text{out}}\langle f(p_1')\bar{f}(p_2')|\, f(p_1)\,\bar{f}(p_2)\rangle^{\text{in}}$$

$$=\delta(p_1'^1-p_1^1)\,\delta(p_2'^1-p_2^1)\,t_{f\bar{f}}(\theta)$$

$$-\,\delta(p_1'^1-p_2^1)\,\delta(p_2'^1-p_1^1)\,r_{f\bar{f}}(\theta).$$

Let $t_{ff}(\theta)$ be the particle-particle scattering amplitude, then crossing yields the relations

$$t_{f\bar{f}}(\theta)=r_{f\bar{f}}(\mathrm{i}\pi-\theta),$$

$$t_{f\bar{f}}(\mathrm{i}\pi-\theta)=t_{ff}(\theta)=t_{\bar{f}\bar{f}}(\theta).$$

Unitarity implies in view of the absence of inelastic channels and assumption (a):

$$t_{f\bar{f}}(-\theta)\,t_{f\bar{f}}(\theta)+r_{f\bar{f}}(-\theta)\,r_{f\bar{f}}(\theta)=1 \tag{1a}$$

$$t_{f\bar{f}}(-\theta)\,t_{f\bar{f}}(\theta)+r_{f\bar{f}}(-\theta)\,t_{f\bar{f}}(\theta)=0 \tag{1b}$$

$$t_{ff}(-\theta)\,t_{ff}(\theta)=1 \tag{1c}$$

Note that if $r_{f\bar{f}}(\theta)\equiv 0$, there exists an infinite family of solutions $t_{f\bar{f}}(\theta)$ to these eqs. (1a, b, c). For $r_{f\bar{f}}(\theta)\neq 0$ it is advantageous to introduce the ratio

$$h(\theta)=\frac{t_{f\bar{f}}(\theta)}{r_{f\bar{f}}(\theta)}$$

which is an odd function of θ due to eq. (1b) and connected to $t_{f\bar{f}}(\theta)$ by

$$t_{f\bar{f}}(\theta)\,t_{ff}(\mathrm{i}\pi+\theta)=\frac{h(\theta)}{h(\mathrm{i}\pi-\theta)} \tag{2}$$

and obeys a quadratic functional equation

$$h(\mathrm{i}\pi-\theta)\,h(\mathrm{i}\pi+\theta)+h^2(\theta)=1. \tag{3}$$

Not all solutions of eq. (3) are compatible with the factorized form of the S-matrix. The compatible ones must satisfy the following conditions [5]

$$S_{ij}S_{ik}S_{jk}=S_{jk}S_{ik}S_{ij} \tag{4}$$

where i, j, k are pairwise unequal and S_{ij} is defined as follows. In a n-body scattering process, there are 2^n

possible configurations of particles (antiparticles) with a fixed set of momenta. Now S_{ij} is a 2^n-dimensional matrix connecting configurations in which only the scattering of particle (antiparticle) with momenta p_i and p_j is considered. In particular, from the process $f(p_1)\,f(p_2)\,\bar{f}(p_3) \to f(p_1)\,\bar{f}(p_2)\,f(p_3)$ eq. (4) yields the additional functional relation:

$$h(\theta_{31}+\theta_{12}) = h(i\pi+\theta_{31})\,h(\theta_{12}) + h(\theta_{31})\,h(i\pi-\theta_{12}), \tag{5a}$$

from which one derives after proper symmetrization:

$$\frac{h(i\pi+\theta_{31}) - h(i\pi-\theta_{31})}{h(\theta_{31})} = \frac{h(i\pi+\theta_{12}) - h(i\pi-\theta_{12})}{h(\theta_{12})} = \text{const.} \tag{5b}$$

We shall set the constant equal to $2\cos\mu$ for convenience. The common solution to eq. (3) and (5a, b) which is an odd function can be uniquely shown to be

$$h(\theta) = \frac{\sinh(r/\pi)\theta}{\sinh i\mu}. \tag{6}$$

Uniqueness of this solution can be easily seen from the observation that $F(x) = h(i\pi - x) + e^{i\mu}\,h(x)$ satisfies the functional equation $F(x+y) = F(x)\cdot F(y)$. Observe that a class of more general solutions of (3) is of the type

$$h(\theta) = \frac{\sinh(\theta\, f(\theta))}{\sinh(i\pi\, f(\theta))}$$

where $f(\theta)$ is an arbitrary even function of period $i\pi$ [6].

It remains to show that all the amplitudes can be obtained from $h(\theta)$. We make two technical assumptions

(i) There exists a value of μ such that $t_{f\bar{f}}(\theta)$ is analytic and non-zero on the physical strip; i.e. $0 < \text{Im}\,\theta < \pi$.

(ii) The function $\{1/(\sinh(z-\theta))\}\ \ln t_{f\bar{f}}(\theta)$ vanishes in absolute value as $|\text{Re}\,z| \to \infty$.

Then Cauchy's formula yields

$$\ln t_{f\bar{f}}(\theta) = \frac{1}{2\pi i}\int_C \frac{dz}{\sinh(z-\theta)}\ \ln t_{f\bar{f}}(z)$$

C being the contour enclosing the physical strip. Since $(1/2i)\ln t_{f\bar{f}}(\theta)$ is the phase shift $\delta_{f\bar{f}}(\theta)$, eq. (2) and (6) yield

$$\delta_{f\bar{f}}(\theta) = -\frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{dz}{\sinh(z-\theta)}\ \ln \frac{\sinh(\mu/\pi)\theta}{\sinh(\mu/\pi)(i\pi-\theta)}$$

or alternatively through Fourier transformations and partial integration:

$$\delta_{f\bar{f}}(\theta) = \frac{1}{2}\int_0^{\infty} \frac{dx}{x}\,\frac{\sin(x/\pi)(i\pi-\theta)\,\sinh(x/2)(\pi/\mu-1)}{\sinh(\pi/\mu)(x/2)\,\cosh(x/2)} \tag{7}$$

Setting $\lambda = \mu/\pi$ and using Malmstén's formula which gives the integral representation of $\log\Gamma(z)$ one can derive consequently the Zamolodchikov's solution [7] from eq. (7). Moreover let us point out that assumption (i) can be interpreted as follows: the coupling constant λ is related to be Thirring coupling constant g by

$$\lambda = 1 + \frac{2g}{\pi}$$

For $\mu < \pi$ we have the repulsive potential ($g < 0$), therefore, there are no bound states.

Finally, we remark that the complete S-matrix which includes bound state scatterings has been treated in [5].

We would like to thank B. Berg, S. Meyer, B. Schroer and R. Seiler for fruitful discussions.

[1] B. Schroer, T.T. Truong and P. Weisz, Phys. Lett. 63B (1976) 422.

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cf. also: B. Berg, The massive Thirring model: Particle scattering in perturbation theory, FU Berlin preprint 77/5.

[3] cf. e.g. R. Flume and S. Meyer, Renormalization of a higher conservation law in the massive thirring model, CERN preprint TH 2243 (1976); Phys. Lett., to be published.
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[4] A.B. Zamolodchikov, Exact S-matrix of quantum sine Gordon solitons, Moscow preprint ITEP-148 (1976).

[5] M. Karowski and H.J. Thun, Complete S-matrix of the massive Thirring model, FU Berlin preprint 77/6.

[6] We thank B. Berg for pointing out this class of solutions to us.

[7] P.H. Weisz, Perturbation theory checks of a proposed exact Thirring model S-matrix, FU Berlin preprint 77/1; Nucl. Phys., to appear. Note that the parameter λ of this reference is the inverse of ours.