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COMPLETE S-MATRIX OF THE MASSIVE THIRRING MODEL

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On the basis of Zamolodchikov's S-matrix for two sine-Gordon solitons we derive the S-matrix for the scattering of an arbitrary number of particles including bound states.

1. Introduction

Since Coleman's [1] work on the equivalence of the massive Thirring model and the sine-Gordon theory there has been some progress in solving this model exactly [2]. The infinite number of conservation laws [3] which forbid particle production and imply the conservation of the set of the particle momenta impose strong restrictions on the S-matrix. One believes [4] that it factorizes into two-particle S-matrices.

The exact two-particle S-matrix for the scattering of solitons (antisolitons) was given recently by Zamolodchikov [5]. On the basis of his results we construct the complete S-matrix for the scattering of an arbitrary number of solitons, antisolitons, and bound states (breathers).

In sect. 2 we generalize the notion of the factorization of the S-matrix to the case of the scattering of an arbitrary number of different kinds of particles. We derive necessary conditions for the two-particle scattering operator. In sect. 3 we review Zamolodchikov's S-matrix for the scattering of two solitons (antisolitons), extend it to the multisoliton sector and prove the above mentioned conditions. After giving a recipe for the S-operator for bound states we calculate in sect. 4 the amplitudes for soliton-breather scattering and for breather-breather scattering. We also present some consistency checks.

2. Factorization of the S-matrix

The infinite sequence of conservation laws [3] ★

★ For the quantized massive Thirring model so far only the first non-trivial conservation law has been established rigorously.

$$\partial_\mu J_k^\mu = 0\ , \qquad k = 1, 3, 5, \ldots\ , \tag{1}$$

corresponding to the conservation of the sum of powers of the momenta imply
(i) absence of particle production,
(ii) equality of the sets of initial and final momenta

$$\{p'_1, \ldots, p'_{n'}\} = \{p_1, \ldots, p_n\}\ .$$

One believes [4] that (i) and (ii) imply the factorization of the S-matrix. In the case of identical particles the factorization can be easily formulated [6].

$${}^{\text{out}}\langle p'_1 \ldots p'_{n'} | p_1 \ldots p_n\rangle^{\text{in}} = {}^{\text{in}}\langle p'_1 \ldots p'_{n'} | p_1 \ldots p_n\rangle^{\text{in}} \prod_{1 \leqslant i < j \leqslant n} e^{2i\delta(p_i, p_j)}\ , \tag{2}$$

where $\delta(p_i, p_j)$ is the two-particle phase shift. Let us now generalize eq. (2) to the case of different kinds of particles $\alpha \in$ A. In the sine-Gordon theory or the massive Thirring model we have A = $\{\mathrm{f}, \bar{\mathrm{f}}, \mathrm{b}_1, \mathrm{b}_2, \ldots\}$ standing for soliton, antisoliton, and breathers. Whenever the properties (i) and (ii) hold true the Fock space decomposes into spaces of definite particle number and fixed set of momenta

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \int \frac{\mathrm{d}p_1^1}{2\omega_1} \cdots \frac{\mathrm{d}p_n^1}{2\omega_n}\, \mathcal{H}_{p_1 \ldots p_n}\ , \tag{3a}$$

where

$$\mathcal{H}_{p_1 \ldots p_n} = \bigoplus_{\alpha_i \in A} (\mathcal{H}_{p_1}^{(\alpha_1)} \otimes \ldots \otimes \mathcal{H}_{p_n}^{(\alpha_n)})'\ , \tag{3b}$$

and $\mathcal{H}_p^{(\alpha)}$ is the (one-dimensional) space of a particle of type α with momentum p. The prime in eq. (3b) denotes symmetrization (antisymmetrization) with respect to bosons (fermions). The space $\mathcal{H}_{p_1 \ldots p_n}$ is left-invariant by the S-operator. Moreover, it is known [6] that the conservation laws (1) imply the numbers n_i of particles of the same type α_i to be unchanged. Therefore, $\mathcal{H}_{p_1 \ldots p_n}$ is built up by smaller subspaces corresponding to fixed particle configurations $\{n_i\}$ with $\Sigma n_i = n$. We do not attempt to express this fact in a more complicated notation.

We characterize a state in $\mathcal{H}_{p_1 \ldots p_n}$ by

$$|\alpha_Q\rangle = |\alpha_{Q_1} \ldots \alpha_{Q_n}\rangle\ ,$$

where $Q = \{Q_1, Q_2, \ldots, Q_n\}$ is a permutation of the integers $\{1, 2, \ldots, n\}$ and we adopt the convention that the particle corresponding to the entry k in $|\alpha_Q\rangle$ has momentum p_k. The states are normalized according to

$$\langle \alpha'_{Q'} | \alpha_Q \rangle = \delta_{n'n} \prod_{k=1}^{n} \delta_{\alpha'_{Q'_k} \alpha_{Q_k}}\ .$$

It is clear from our definition of states that the S-operator acts on them by a possible permutation of the labels α_{Q_k} in $|\alpha_Q\rangle$. The factorization of the S-matrix can now be formulated for the case of different particles, too,

$$ {}^{\mathrm{out}}\langle\alpha'_{Q'}|\alpha_Q\rangle^{\mathrm{in}} = \langle\alpha'_{Q'}| \prod_{i\leqslant i<j\leqslant n} S(p_i, p_j)|\alpha_Q\rangle\,. \tag{4} $$

Since the two-particle S-matrices $S(p_i, p_j)$, in general, fail to commute it is necessary to specify the order of the factors occuring in eq. (4). We take the following "time-ordering prescription": draw in a space time diagram lines parallel to the momenta (wave vectors) $p_1, p_2, ..., p_n$ and assign an interaction time t_{ij} $(i<j)$ to each intersection point, cf. fig. 2a. Then the factors in eq. (4) are ordered according to their times t_{ij}, i.e. the operator with the lowest time acting first on the state $|\alpha_Q\rangle$. Obviously, the ordering of the interaction times depends on the actual choice of momentum lines. In order to have an unambiguous definition it is necessary that the product of the two-particle S-matrices $S_{ij} = S(p_i, p_j)$ is invariant under arbitrary parallel shifts of the momentum lines. This amounts to two necessary and sufficient commutation relations

$$ S_{ij}S_{kl} = S_{kl}S_{ij} \qquad \text{for } i, j, k, l \text{ all unequal}\,, \tag{5a} $$

$$ S_{ij}S_{ik}S_{jk} = S_{jk}S_{ik}S_{ij} \qquad \text{for } i, j, k \text{ all unequal}\,, \tag{5b} $$

which guarantee the product of the two-particle S-matrices in eq. (4) to be independent (a) of the relative ordering of the intersection points of two different pairs of lines and (b) of the parallel shift of any line beyond the intersection point of two other lines, cf. fig. 2b. For the case of the sine-Gordon theory eqs. (5a, b) are proved in sect. 3. Note that in the case $p_1 > p_2 > ... > p_n$ a possible "canonical" ordering of the factors in eq. (4) is given by the sequence

$$ (S_{12} \ldots S_{1n})(S_{23} \ldots S_{2n}) \ldots S_{n-2\,n}\,. $$

Finally, we remark that a structure of the S-matrix which resembles eq. (4) and commutation relations similar to eqs. (5) are known from the many-body δ-function potential scattering [7].

The two-particle S-matrices S_{ij}, in general, have both a transmission and a reflection part

$$ S_{ij} = T_{ij} + R_{ij}\,, \tag{6} $$

defined by

$$ T_{ij}|\alpha_Q\rangle = t_{\alpha_{Q_i}\alpha_{Q_j}}(p_i, p_j)|\alpha_Q\rangle\,, \tag{6a} $$

$$ R_{ij}|\alpha_Q\rangle = r_{\alpha_{Q_i}\alpha_{Q_j}}(p_i, p_j)|\alpha_{Q(i,j)}\rangle\,, \tag{6b} $$

where (i, j) in (6b) is the transposition interchanging the indices i and j, i.e.

$$ |\alpha_Q\rangle = |\ldots \alpha_{Q_i} \ldots \alpha_{Q_j} \ldots\rangle\,, $$

$$ |\alpha_{Q(i,j)}\rangle = |\ldots \alpha_{Q_j} \ldots \alpha_{Q_i} \ldots\rangle\,. $$

In the case of only two types of particles which are each others antiparticles, say solitons f and antisolitons $\bar{\mathrm{f}}$, we have four functions occuring in eq. (6), t_{ff}, $t_{\bar{\mathrm{f}}\bar{\mathrm{f}}}$, $t_{\mathrm{f}\bar{\mathrm{f}}}$, and $r_{\mathrm{f}\bar{\mathrm{f}}}$. They are, however, related by crossing symmetry and unitarity. Let us introduce the rapidity Θ by

$$p_i^0 = m_i \operatorname{ch}\Theta_i \,, \qquad p_i^1 = m_i \operatorname{sh}\Theta_i \,,$$

i.e.

$$(p_i + p_j)^2 = m_i^2 + m_j^2 + 2m_i m_j \operatorname{ch}\Theta_{ij} \,,$$

$$(p_i - p_j)^2 = m_i^2 + m_j^2 + 2m_i m_j \operatorname{ch}(i\pi - \Theta_{ij}) \,,$$

where

$$\Theta_{ij} = \Theta_i - \Theta_j \,,$$

is the rapidity difference. The relation between the Mandelstam variable s and the rapidity Θ in the various regions of interest is given in fig. 1.

The crossing relations

$$\langle \mathrm{f}\bar{\mathrm{f}} | S(p_1, p_2) | \mathrm{f}\bar{\mathrm{f}} \rangle = \langle \mathrm{ff} | S(p_1, -p_2) | \mathrm{ff} \rangle = \langle \bar{\mathrm{f}}\,\bar{\mathrm{f}} | S(-p_1, p_2) | \bar{\mathrm{f}}\bar{\mathrm{f}} \rangle \,,$$

$$\langle \bar{\mathrm{f}}\mathrm{f} | S(p_1, p_2) | \mathrm{f}\bar{\mathrm{f}} \rangle = \langle \bar{\mathrm{f}}\mathrm{f} | S(p_1, -p_2) | \mathrm{f}\bar{\mathrm{f}} \rangle \,,$$

then read

$$t(\Theta) = u(i\pi - \Theta) \,, \tag{7a}$$

$$r(\Theta) = r(i\pi - \Theta) \,, \tag{7b}$$

where

$$t = t_{\mathrm{f}\bar{\mathrm{f}}} \,, \qquad r = r_{\mathrm{f}\bar{\mathrm{f}}} \,, \qquad u = t_{\mathrm{ff}} = t_{\bar{\mathrm{f}}\bar{\mathrm{f}}} \,.$$

From the unitarity of the S-matrix one deduces the relations

$$t(-\Theta)\, t(\Theta) + r(-\Theta)\, r(\Theta) = 1 \,, \tag{8a}$$

$$t(-\Theta)\, r(\Theta) + r(-\Theta)\, t(\Theta) = 0 \,, \tag{8b}$$

$$t(i\pi + \Theta)\, t(i\pi - \Theta) = 1 \,, \tag{8c}$$

which lead to

$$r^2(\Theta) = t^2(\Theta)[1 - (t(-\Theta)\, t(\Theta))^{-1}] \,. \tag{9}$$

From eqs. (7) and (8) we see that, in fact, there is only one independent function, since both the scattering amplitude for identical particles $u(\Theta)$ and the reflection amplitude $r(\Theta)$ may be expressed in terms of the particle-antiparticle transmission amplitude $t(\Theta)$.

Fig. 1. Relation between the Mandelstam variable s and the rapidity Θ. The physical sheet is mapped in the complex Θ-plane into the strip $0 \leqslant \mathrm{Im}\,\Theta \leqslant \pi$. The regions $s \geqslant s_{th}$ (s-channel) and $s \leqslant 0$ (u-channel) correspond to $\mathrm{Re}\,\Theta \geqslant 0$, $\mathrm{Im}\,\Theta = 0$ and $\mathrm{Re}\,\Theta \leqslant 0$, $\mathrm{Im}\,\Theta = \pi$, respectively. The points 1, 2, ... denote the position of possible poles (bound states).

3. Solitons

In this section we first review some known results on the two-particle S-matrix of the sine-Gordon theory for the scattering of solitons (antisolitons). Then we prove the commutation relations (5) for this S-matrix. Finally, we discuss the poles of this S-matrix which correspond to soliton-antisoliton bound states (breathers).

For special cases of the coupling constant, namely if the quantity

$$\lambda = 1 + \frac{2g}{\pi} = \frac{8\pi}{\beta^2}\left(1 - \frac{\beta^2}{8\pi}\right) = \frac{8\pi}{\gamma},$$

is equal to an integer, Korepin and Faddeev [6] gave an explicit formula for the transmission amplitude $t(\Theta)$ for soliton-antisoliton scattering,

$$t(\Theta) = (-1)^\lambda \prod_{k=1}^{\lambda} \frac{e^{\Theta - i\pi k/\lambda} + 1}{e^{\Theta} + e^{-i\pi k/\lambda}}, \tag{10}$$

which has poles at

$$\Theta_k = i\pi(1 - k/\lambda), \tag{11}$$

corresponding to the quasiclassical sine-Gordon spectrum [9]

$$m_k = 2m \sin \pi \frac{k}{2\lambda}. \tag{12}$$

One easily verifies that, in this case (λ integer), the reflection amplitude $r(\Theta)$ vanishes identically.

It is convenient to write eq. (10) in the form

$$t(\Theta) \equiv t(\varphi) = F(\varphi)/F(2 - \varphi), \tag{13a}$$

where $\Theta = i\pi\varphi$ and

$$F(\varphi) = \prod_{k=1}^{\lambda-1} \prod_{l=0}^{\infty} \frac{(2l + k/\lambda + \varphi)}{(2l - 1 + k/\lambda + \varphi)}$$

$$= \prod_{k=0}^{\infty} \prod_{l=0}^{\infty} \frac{(2l + (k+1)/\lambda + \varphi)(2l + k/\lambda + \varphi)}{(2l - 1 + (k+1)/\lambda + \varphi)(2l + 1 + k/\lambda + \varphi)}. \tag{13b}$$

The last line of eq. (13b) is defined for arbitrary λ. The functions $r(\varphi)$ and $u(\varphi)$ can be calculated from eqs. (7), (8) and (13)

$$r(\varphi) = \frac{\sin \pi\lambda}{\sin \pi\lambda\varphi} t(\varphi) , \tag{14a}$$

$$u(\varphi) = \frac{\sin \pi\lambda(1 - \varphi)}{\sin \pi\lambda\varphi} t(\varphi) . \tag{14b}$$

The analytic interpolation of the transmission amplitude $t(\varphi)$ defined by eq. (13) was given by Zamolodchikov [5]. It satisfies the crossing and unitarity conditions (7) and (8), and it is in agreement with known results [10] of perturbative calculations in the Thirring model coupling constant g and in the quasiclassical limit $\beta \to 0$ [6,8].

We now prove the commutation relations (5a, b) which guarantee an unambiguous definition of the complete S-matrix in terms of Zamolodchikov's two-particle S-matrices. First of all, eq. (5a) trivially holds since for unequal i, j, k, l the possible transpositions of the particle labels, cf. eqs. (6a, b), do not interfere. So we are left with the proof of eq. (5b). Using eqs. (6) and (14) we calculate a specific matrix element, cf. fig. 2, with $i = 1, j = 2, k = 3$,

$$\begin{aligned}
&\langle \mathrm{f}\bar{\mathrm{f}}\mathrm{f}|(S_{12}S_{13}S_{23} - S_{23}S_{13}S_{12})|\mathrm{f}\mathrm{f}\bar{\mathrm{f}}\rangle \\
&= r(\varphi_{12})\, r(\varphi_{13})\, t(\varphi_{23}) + t(\varphi_{12})\, u(\varphi_{13})\, r(\varphi_{23}) - r(\varphi_{23})\, t(\varphi_{13})\, u(\varphi_{12}) \\
&= \frac{t(\varphi_{12})}{\sin \pi\lambda\varphi_{12}} \frac{t(\varphi_{13})}{\sin \pi\lambda\varphi_{13}} \frac{t(\varphi_{23})}{\sin \pi\lambda\varphi_{23}} \\
&\times \{\sin \pi\lambda\varphi_{23} \sin \pi\lambda + \sin \pi\lambda\varphi_{12} \sin \pi\lambda(1 - \varphi_{13}) - \sin \pi\lambda\varphi_{13} \sin \pi\lambda(1 - \varphi_{12})\} \\
&= 0 .
\end{aligned}$$

The last equality sign holds since the variables φ_{ij} are difference variables, i.e.

$$\varphi_{12} + \varphi_{23} = \varphi_{13} .$$

By similar elementary calculations all other matrix elements of $S_{12}S_{13}S_{23}$ and $S_{23}S_{13}S_{12}$ can be shown to be equal.

In order to discuss the bound states we consider the amplitudes

$$s_\pm = t \pm r ,$$

which describe the scattering of soliton-antisoliton states of definite charge conjugation and space reflection parity

$$C = P = \pm 1 .$$

From eqs. (13) and (14) it is seen that they have poles at

$$\varphi_k = 1 - \frac{k}{\lambda}, \qquad k = 1, 2, \dots , \tag{15}$$

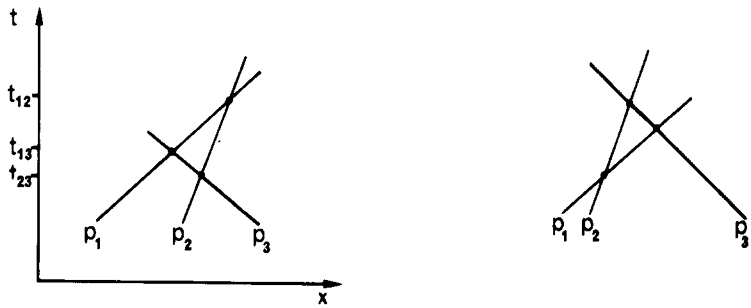

Fig. 2. (a) Time ordering of the interaction of a three-particle scattering process. (b) Different time ordering obtained by parallel shifting a line.

which correspond to bound states if φ_k lies in the physical sheet, i.e. $k < \lambda$. Their masses m_k are given by the sine-Gordon spectrum (12). More precisely, $s_+ (s_-)$ has poles at φ_k for even (odd) values of k only, and

$$\operatorname*{Res}_{s=m_k^2} s_+ = R_k > 0 \,, \qquad \text{for } k \text{ even} \,,$$

$$\operatorname*{Res}_{s=m_k^2} s_- = R_k < 0 \,, \qquad \text{for } k \text{ odd} \,.$$

This is consistent with the opinion that the lowest bound state ($k = 1$) is a pseudoscalar particle which corresponds to the (elementary) sine-Gordon field, since, in general, poles of the S-matrix corresponding to states with symmetric (antisymmetric) wave functions must have a negative (positive) residue, and fermion-antifermion bound states with symmetric (antisymmetric) wave functions have negative (positive) C and P parity.

The amplitude $u(\varphi)$ is continuous at $\varphi = \varphi_k$, therefore we conclude from the above considerations that the residue of the soliton-antisoliton S-operator at $s_{12} = (p_1 + p_2)^2 = m_k^2$ acts as a projection operator, i.e.

$$P_{12}^k := \frac{1}{R_k} \operatorname*{Res}_{s_{12}=m_k^2} S(p_1, p_2) \,, \tag{16}$$

projects onto chargeless states

$$\sqrt{\tfrac{1}{2}}(|f\bar{f}\rangle + (-1)^k |\bar{f}f\rangle) \,,$$

with momentum $p = p_1 + p_2$, mass m_k, and parity $(-1)^k$. We identify them with the bound states (breathers) b_k.

4. Bound states

In this section we calculate the amplitudes of soliton-breather scattering and of breather-breather scattering. We also present several consistency checks.

From eqs. (5b) and (16) we find, cf. fig. 2,

$$\frac{1}{R_n} \operatorname*{Res}_{s_{12}=m_n^2} S_{12}S_{13}S_{23} = P_{12}^n S_{13}S_{23} = S_{23}S_{13}P_{12}^n$$

$$= S(p_1 + p_2, p_3)\Big|_{\substack{(p_1+p_2)^2=m_n^2 \\ p_3^2=m^2}} . \tag{17}$$

The last equation serves as a definition of the two-particle S-matrix in the $\mathrm{b}_n\mathrm{f}$ and $\mathrm{b}_n\bar{\mathrm{f}}$ sector. Obviously, there is no reflection since the projector P_{12}^n appears on both sides in eq. (17). The formula (17) may be interpreted diagrammatically as follows, cf. fig. 3, if the lines corresponding to the momenta p_1 and p_2 of f and $\bar{\mathrm{f}}$ in a state of definite parity $(-1)^n$ become "parallel" then the two-particle S-operator is replaced by a projection operator on a boson state b_n of momentum $p_1 + p_2$. More precisely, "parallel" means $\mathrm{Re}\, p_1 = \mathrm{Re}\, p_2$ which is a consequence of the conditions $(p_1 + p_2)^2 = m_n^2$ and $p_1 + p_2 =$ real.

For the scattering of two bosons b_n and b_m we obtain similarly, cf. fig. 4,

$$\frac{1}{R_n} \operatorname*{Res}_{s_{12}=m_n^2} \frac{1}{R_m} \operatorname*{Res}_{s_{34}=m_m^2} S_{12}S_{13}S_{14}S_{23}S_{24}S_{34}$$

$$= P_{12}^n P_{34}^m S_{14}S_{13}S_{24}S_{23} = S_{23}S_{13}S_{24}S_{14}P_{12}^n P_{34}^m$$

$$= S(p_1 + p_2, p_3 + p_4)\Big|_{\substack{(p_1+p_2)^2=m_n^2 \\ (p_3+p_4)^2=m_m^2}} . \tag{18}$$

By the same argument as above there is no reflection for boson-boson scattering. The diagrammatical interpretation of eq. (18) may be read off fig. 5.

Let us now calculate the boson-fermion scattering amplitude $t_{\mathrm{b}_n\mathrm{f}}(\varphi)$. From eqs. (16), (17) and (6) we have

$$t_{\mathrm{b}_n\mathrm{f}}(\varphi) \equiv t_{\mathrm{b}_n\mathrm{f}}(p_1 + p_2, p_3) \equiv \langle \mathrm{b}_n\mathrm{f}|S(p_1 + p_2, p_3)|\mathrm{b}_n\mathrm{f}\rangle$$

$$= \tfrac{1}{2}[t_{23}u_{13} + (-1)^n r_{23}r_{13} + u_{23}t_{13}] , \tag{19}$$

where

$$t_{23} = t(\varphi_{23}) , \qquad \text{etc.}$$

and

$$\varphi_{13} = \varphi + \tfrac{1}{2}\left(1 - \frac{n}{\lambda}\right) ,$$

$$\varphi_{23} = \varphi - \tfrac{1}{2}\left(1 - \frac{n}{\lambda}\right) .$$

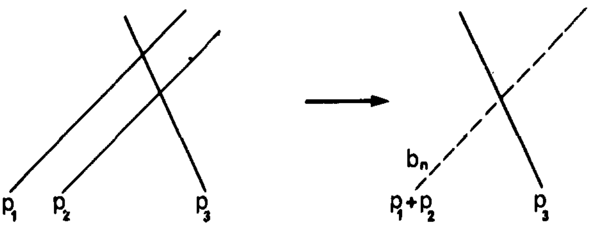

Fig. 3. Composition of a fermion-antifermion pair $\mathrm{f\bar{f}}$ of parity $(-1)^n$ with momenta p_1, p_2 to a boson b_n with momentum $p_1 + p_2$.

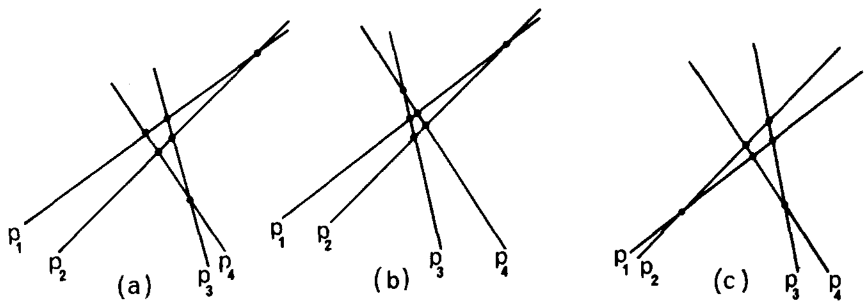

Fig. 4. Three different time orderings for four-particle scattering.

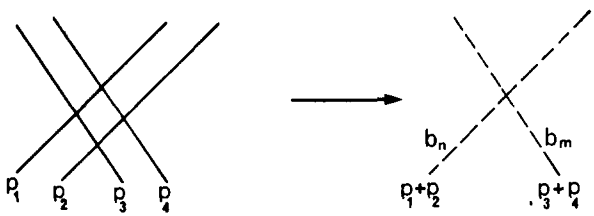

Fig. 5. Composition of two $\mathrm{f\bar{f}}$ pairs to two bosons b_n, b_m.

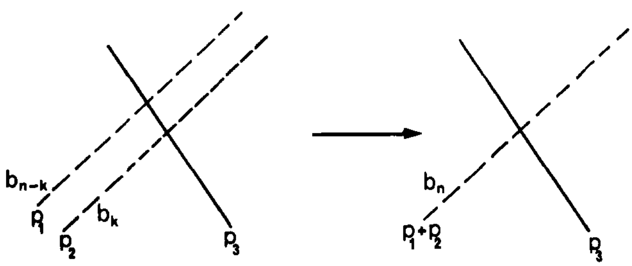

Fig. 6. Composition of two bosons b_{n-k}, b_k to a boson b_n in boson-fermion scattering.

Using eqs. (13) and (14) for u, r and t we find after a lengthy calculation

$$t_{\mathrm{b}_n\mathrm{f}}(\varphi) = (-1)^n \prod_{r=1}^{n-1} \frac{\sin \frac{1}{2}\pi(\varphi + (n-2r)/2\lambda + \frac{1}{2})}{\sin \frac{1}{2}\pi(\varphi - (n-2r)/2\lambda - \frac{1}{2})}$$

$$\times \prod_{r=0}^{n} \frac{\sin \frac{1}{2}\pi(\varphi + (n-2r)/2\lambda + \frac{1}{2})}{\sin \frac{1}{2}\pi(\varphi - (n-2r)/2\lambda - \frac{1}{2})} . \tag{20}$$

Similarly the boson-boson amplitude $t_{\mathrm{b}_n\mathrm{b}_m}(\varphi)$ is obtained from eqs. (18), (16) and (6)

$$t_{\mathrm{b}_n\mathrm{b}_m}(\varphi) = \tfrac{1}{2}\,[t_{14}u_{13}u_{24}t_{23} + r_{14}r_{13}r_{24}r_{23} + u_{14}t_{13}t_{24}u_{23}$$

$$+ (-1)^n (r_{14}t_{13}r_{24}u_{23} + u_{14}r_{13}t_{24}r_{23})$$

$$+ (-1)^m (r_{14}r_{13}t_{24}u_{23} + u_{14}t_{13}r_{24}r_{23})$$

$$+ (-1)^{n+m} (r_{14}t_{13}t_{24}r_{23} + u_{14}r_{13}r_{24}u_{23})] , \tag{21}$$

where

$$\varphi_{14} = \varphi + 1 - \frac{n+m}{2\lambda} , \qquad \varphi_{23} = \varphi - 1 + \frac{n+m}{2\lambda} ,$$

$$\varphi_{13} = \varphi - \frac{n-m}{2\lambda} , \qquad \varphi_{24} = \varphi + \frac{n-m}{2\lambda} .$$

One finds

$$t_{\mathrm{b}_n\mathrm{b}_m}(\varphi) = \frac{\mathrm{tg}\, \frac{1}{2}\pi(\varphi + (n+m)/2\lambda)}{\mathrm{tg}\, \frac{1}{2}\pi(\varphi - (n+m)/2\lambda)} \prod_{r=1}^{n-1} \frac{\mathrm{tg}\, \frac{1}{2}\pi(\varphi + (n+m-2r)/2\lambda)}{\mathrm{tg}\, \frac{1}{2}\pi(\varphi - (n+m-2r)/2\lambda)}$$

$$\times \prod_{r=1}^{m-1} \frac{\mathrm{tg}\, \frac{1}{2}\pi(\varphi + (n+m-2r)/2\lambda)}{\mathrm{tg}\, \frac{1}{2}\pi(\varphi - (n+m-2r)/2\lambda)} . \tag{22}$$

For integer values of λ [6] and in the quasiclassical limit including first quantum corrections [14] eqs. (19)–(22) reduce to known results. In particular, the scattering amplitude for two elementary sine-Gordon bosons b_1 is given by

$$t_{\mathrm{b}_1\mathrm{b}_1}(\varphi) = \frac{\mathrm{tg}\, \frac{1}{2}\pi(\varphi + 1/\lambda)}{\mathrm{tg}\, \frac{1}{2}\pi(\varphi - 1/\lambda)} = \frac{\sin \pi\varphi + \sin \pi/\lambda}{\sin \pi\varphi - \sin \pi/\lambda} , \tag{23}$$

in agreement with the general structure quoted in [6,11]

$$t_{\mathrm{b}_1\mathrm{b}_1}(\Theta) = \frac{8g(\mathrm{ch}\,\Theta)\,\mathrm{sh}\,\Theta - i}{8g(\mathrm{ch}\,\Theta)\,\mathrm{sh}\,\Theta + i} , \tag{24}$$

if

$$g(\text{ch}\ \Theta) = -[8 \sin \pi/\lambda]^{-1} ,$$

which is consistent with perturbative calculations. One easily verifies that all amplitudes $t_{b_n b_m}$ can be cast into the form (24) with, however, more complicated functions $g(\cos \Theta)$.

The pole of $t_{b_n b_m}$ at $\varphi = (n+m)/2\lambda$ or $s = m_{n+m}^2$ corresponds to the bound state b_{n+m} if $n + m < \lambda$. One easily derives the following relations which are based on charge conjugation, crossing symmetry, and unitarity

$$t_{b_n f}(\varphi) = t_{b_n \bar{f}}(\varphi) , \tag{25a}$$

$$t_{b_n f}(\varphi) = t_{b_n \bar{f}}(1 - \varphi) , \tag{25b}$$

$$t_{b_n f}(-\varphi)\, t_{b_n f}(\varphi) = 1 , \tag{25c}$$

$$t_{b_n b_m}(\varphi) = t_{b_n b_m}(1 - \varphi) , \tag{25d}$$

$$t_{b_n b_m}(-\varphi)\, t_{b_n b_m}(\varphi) = 1 . \tag{25e}$$

In order to show that our scheme is consistent we generalize the meaning of eqs. (16) and (17): The prescriptions remain true if they are applied to two bound states b_{n-k}, b_k at $(p_1 + p_2)^2 = m_n^2$. By the same methods as in the derivation of eq. (20) one verifies, cf. fig. 6, the relation

$$t_{b_{n-k} f}(p_1, p_3)\, t_{b_k f}(p_2, p_3)|_{(p_1+p_2)^2 = m_n^2} = t_{b_n f}(p_1 + p_2, p_3) . \tag{26}$$

Comparing eqs. (19) and (26) we find the result: the construction of the two-particle S-matrix element of a boson b_n and a fermion f by the prescription of taking the residue of a three-particle S-matrix element is independent of whether the boson b_n is considered as a bound state of a fermion-antifermion pair $f\bar{f}$, cf. fig. 3, or as a bound state of lower bosons b_{n-k}, b_k, cf. fig. 6. More generally, a boson b_n may be considered as bound state composed of $b_{n-k} b_{k-l} b_{l-j}$ ....

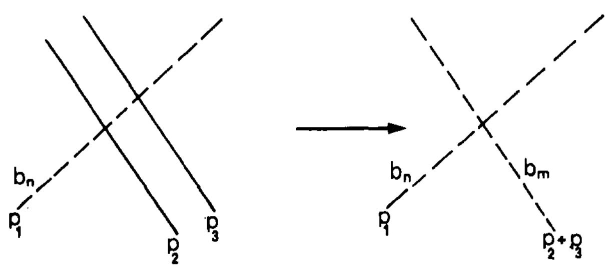

Fig. 7. Composition of a fermion-antifermion pair $f\bar{f}$ to a boson b_m in boson-fermion scattering.

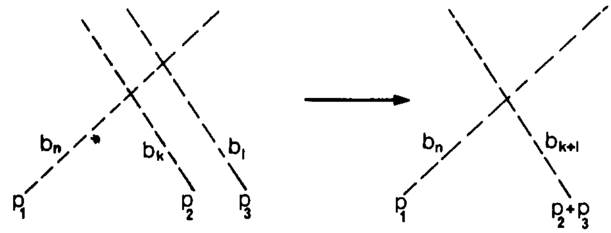

Fig. 8. Composition of two bosons b_k, b_l to a boson b_{k+l} in boson-boson scattering.

A second consistency check is given by the relation, cf. fig. 7,

$$t_{b_n f}(p_1, p_2)\, t_{b_n \bar{f}}(p_1, p_3)\Big|_{(p_2+p_3)^2 = m_m^2} = t_{b_n b_m}(p_1, p_2 + p_3)\,, \tag{27}$$

which is in agreement with eqs. (20), (22).

Thirdly one has, cf. fig. 8,

$$t_{b_n b_k}(p_1, p_2)\, t_{b_n b_l}(p_1, p_3)\Big|_{(p_2+p_3)^2 = m_{k+l}^2} = t_{b_n b_{k+l}}(p_1, p_2 + p_3)\,. \tag{28}$$

One also observes the relations

$$t_{b_n f}(\varphi)|_{\lambda = n} = t_{\bar{f} f}(\varphi)\, t_{ff}(\varphi)|_{\lambda = n}\,, \tag{29}$$

$$t_{b_n b_m}(\varphi)|_{\lambda = n} = t_{b_m \bar{f}}(\varphi)\, t_{b_m f}(\varphi)|_{\lambda = n}\,, \tag{30}$$

which in view of eq. (15) and figs. 3 and 7 are easily understood: if λ approaches the integer n from above then the pole φ_n leaves the physical sheet and the bound states b_n disintegrate into their constituents f, $\bar{f}$, while b_m remains bound if $m < \lambda = n$.

Finally, we quote some more physical properties of the scattering amplitudes. The high energy behaviour is given by

$$t_{b_n f}(s) \sim (-1)^n\,, \qquad t_{b_n b_m}(s) \sim 1 \qquad \text{for } s \to \infty\,. \tag{31}$$

Near the elastic thresholds (there are no inelastic ones) the amplitudes behave like

$$t_{b_n f}(s) \sim (-1)^n [1 + c_n \sqrt{s - (m + m_n)^2}\,] \qquad \text{for } s \to (m + m_n)^2\,, \tag{32a}$$

$$t_{b_n b_m}(s) \sim [1 + c_{nm} \sqrt{s - (m_n + m_m)^2}\,] \qquad \text{for } s \to (m_n + m_m)^2\,, \tag{32b}$$

where the constants c_n, c_{nm} may be determined from eqs. (20) and (22). The various coupling strengths of a soliton-antisoliton to a bound state and of two bound states to a third one are, respectively,

$$g_{f\bar{f}b_n} = \operatorname*{Res}_{s_{12} = m_n^2} t_{f\bar{f}} = (-1)^n 2m^2 \left(2 \cos \frac{\pi n}{2\lambda} \prod_{r=1}^{n-1} \operatorname{ctg} \frac{\pi r}{2\lambda}\right)^2\,, \tag{33a}$$

$$g_{\mathrm{b}_n\mathrm{b}_m\mathrm{b}_{n+m}} = \operatorname*{Res}_{s_{12}=m_{n+m}^2} t_{\mathrm{b}_n\mathrm{b}_m} = -2m_n m_m \left(2 \sin \frac{\pi(n+m)}{2\lambda}\right)^2$$

$$\times \prod_{r=1}^{n-1} \frac{\mathrm{tg}[(\pi/2\lambda)(n+m-r)]}{\mathrm{tg}(\pi r/2\lambda)} \prod_{r=1}^{m-1} \frac{\mathrm{tg}[(\pi/2\lambda)(n+m-r)]}{\mathrm{tg}(\pi r/2\lambda)} . \tag{33b}$$

From eq. (33a) one directly sees that the coupling $g_{\mathrm{f}\bar{\mathrm{f}}\mathrm{b}_n}$ vanishes in the limit $\lambda \to n$ when the bound state decays.

5. Conclusions

Using Zamolodchikov's two-particle S-matrix for the scattering of solitons and/or antisolitons [5] we have calculated the complete S-matrix of the sine-Gordon theory (massive Thirring model). According to the general discussion of sect. 2 it factorizes into two-particle scattering amplitudes t, r, u; $t_{\mathrm{b}_n\mathrm{f}}$, $t_{\mathrm{b}_n\bar{\mathrm{f}}}$, and $t_{\mathrm{b}_n\mathrm{b}_m}$ which are given by eqs. (13), (14), (20), (22) and (25). From eqs. (20) and (22) it is seen that the boson-fermion amplitudes $t_{\mathrm{b}_n\mathrm{f}}$ and the boson-boson amplitudes $t_{\mathrm{b}_n\mathrm{b}_m}$ in general have "redundant poles" (mostly double poles) also in the physical sheet which are not related to bound states [12]. A similar situation is known from the exactly soluble δ-function potential scattering [7]. In particular, the simple poles in the boson-fermion amplitudes $t_{\mathrm{b}_n\mathrm{f}}$ do not seem to correspond to bound states in the soliton sector. Hence, the massive Thirring model is completely solved on the mass shell. The much more complicated problem of finding the off-shell solution (i.e. the determination of the Wightman functions) remains open.

Finally, we remark that Zamolodchikov's two-particle S-matrix can be derived uniquely from unitarity, crossing symmetry, and factorization in the sense of sect. 2 without resorting to any assumptions on the spectrum. The latter is a consequence of the above mentioned properties [13].

Note added in proof: After having completed this paper we received a preprint by Zamolodchikov [15] in which similar results are derived.

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