# CONSERVED CURRENTS IN THE MASSIVE THIRRING MODEL

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Received 11 June 1976

The existence of an infinite set of conserved currents in the massive Thirring model is discussed. The first four nontrivial currents are given explicitely.

From Coleman's Correspondence [1] of the quantum Sine-Gordon theory and the massive Thirring model the existence of additional conservation laws in the latter has been conjectured [2, 8]. Calculations of (3 → 3)-particle scattering in tree [3] and one loop [4] approximations were reported recently. These results can be proved generally for (n → n)-particle-scattering [5]. One finds the sets of incoming and outgoing particle momenta to be equal, which implies the conservation of particle and antiparticle numbers separately. In the Sine-Gordon theory an infinite set of conserved currents [6] is known to be responsible for analogous properties [7]. Coleman's work suggests a translation of these currents into corresponding ones in the massive Thirring model.

The integrability of the classical Sine-Gordon equation

$$\Box\varphi = -\frac{\alpha}{\beta}\sin\beta\varphi$$

is connected with the existence of these conservation laws, i.e.

$$\partial_\mu j^\mu = \partial_- j_n^- + \partial_+ j_n^+ = 0, \qquad n = 1, 2, 3, \ldots$$

with the notation

$$a^\pm = (1/\sqrt{2})(a^0 \pm a^1) = a_\mp$$

for any two-vector $a^\mu$.

The current components $j_n^-$ are polynomials of $\partial_+\varphi, \ldots, \partial_+^n\varphi$ while the components $j_n^+$ are products of $\cos\varphi$ respectively $\sin\varphi$ times a polynomial of $\partial_+\varphi, \ldots, \partial_+^{n-1}\varphi$. The only term in $j_n^-$ which is relevant in the asymptotic limits $t \to \pm\infty$ is proportional to the bilinear expression

$$\partial_+\varphi\, \partial_+^n\varphi. \tag{1}$$

Since the lightlike charges corresponding to $j_n^-$ are conserved we have

$$\sum_{\mathrm{i}} (p_+^{\mathrm{i}})^n = \sum_{\mathrm{f}} (p_+^{\mathrm{f}})^n, \qquad n = 1, 3, 5, \ldots$$

where $p^{\mathrm{i}}$ and $p^{\mathrm{f}}$ are the momenta of the initial and final particles, respectively. For even values of $n$ the charges vanish identically. Another set of currents is obtained by interchanging $x^+$ and $x^-$. The quantized currents $j_n$ can be defined in terms of normal products. One can apply Coleman's translation formulas for $\partial_+\varphi$, $\cos\varphi$, and $\sin\varphi$ and finally obtain currents in terms of the massive Thirring field.

The field equation of the massive Thirring model reads

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = g\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi, \quad \mu = +, -.$$

With $\gamma^+ = \sqrt{2}\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\gamma^- = \sqrt{2}\begin{pmatrix}0&0\\1&0\end{pmatrix}$ and the substitutions

$$\frac{\sqrt{2}}{m}\partial_{\mp} \to \partial_{\mp}, \qquad \sqrt{\frac{2g}{m}}\,\psi \to \psi$$

we obtain

$$\mathrm{i}\partial_-\psi_1 = \psi_2 + \psi_1\psi_2^*\psi_2, \qquad \mathrm{i}\partial_+\psi_2 = \psi_1 + \psi_2\psi_1^*\psi_1. \tag{2}$$

In this paper we take $\psi$ to be a classical fermion field obeying (2) and the usual anticommutation relations. The bilinear parts of the Thirring currents $j_n^-, j_n^+$ which are obtained from (1) and the equations of motion (2) are, respectively,

$$\mathrm{i}^n\psi_1^*\partial_+^n\psi_1 + \text{h.c.}, \qquad \mathrm{i}^{n-1}\psi_2^*\partial_+^{n-1}\psi_1 + \text{h.c.}$$

For integers $\{a\}$, $\{b\}$ with $a_i < a_{i+1}$, $b_i < b_{i+1}$ we define

$$F(a_l, ..., a_1; b_1, ..., b_r) = \partial_+^{a_l}\psi_1^* \dots \partial_+^{a_1}\psi_1^*\partial_+^{b_1}\psi_1 \dots \partial_+^{b_r}\psi_1.$$

Then the currents may be written in the general form

$$j_n^- = \mathrm{i}^n\psi_1^*\partial_+^n\psi_1 + \sum_{\{a\}\{b\}} c_{\{a\}\{b\}} F(a_k, ..., a_1; b_1, ..., b_k) + \text{h.c.}, \tag{3a}$$

$$j_n^+ = \mathrm{i}^{n-1}\psi_2^*\partial_+^{n-1}\psi_1 + \psi_2^*\left[\sum_{\{a\}\{b\}} d_{\{a\}\{b\}} F(a_{k-1}, ..., a_1; b_1, ..., b_k)\right] + \text{h.c.} \tag{3b}$$

where the summations range over all strictly ordered sets of integers $\{a\}$, $\{b\}$ with the restrictions for (3a):

$$2 \leqslant k \leqslant \sqrt{n+1}, \qquad \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i + k = n = 1, \qquad \sum_{i=1}^{k} a_i \leqslant \sum_{i=1}^{k} b_i, \tag{4a}$$

for (3b):

$$2 \leqslant k \leqslant \tfrac{1}{2}(1 + \sqrt{4n-3}), \qquad \sum_{i=1}^{k-1} a_i + \sum_{i=1}^{k} b_i + k - 1 = n - 1. \tag{4b}$$

The energy-momentum conservation can be expressed by the current

$$j_1^- = \mathrm{i}\psi_1^*\partial_+\psi_1 + \text{h.c.}, \qquad j_1^+ = \psi_2^*\psi_1 + \text{h.c.}$$

The first four nontrivial currents may be written in the form

$$j_3^- = -\mathrm{i}\psi_1^*\partial_+^3\psi_1 + 3F(1,0;0,1) + \text{h.c.}, \qquad j_3^+ = -\psi_2^*\partial_+^2\psi_1 + \mathrm{i}\psi_2^*F(0;0,1) + \text{h.c.}$$

$$j_5^- = \mathrm{i}\psi_1^*\partial_+^5\psi_1 - 19F(1,0;0,3) - 9F(1,0;1,2) - 14F(2,0;0,2) + \text{h.c.}$$

$$j_5^+ = \psi_2^*\partial_+^4\psi_1 - \mathrm{i}\psi_2^*[3F(0;0,3) + 2F(0;1,2) + 7F(1;0,2) + 5F(2;0,1)] + \text{h.c.}$$

$$j_7^- = -\mathrm{i}\psi_1^*\partial_+^7\psi_1 + 36F(1,0;0,5) + 53F(1,0;1,4) + 31F(1,0;2,3)$$
$$+ 82F(2,0;0,4) + 77F(2,0;1,3) + 53F(3,0;0,3) + 46F(3,0;1,2) + \text{h.c.}$$

$$j_7^+ = -\psi_2^*\partial_+^6\psi_1 + \mathrm{i}\psi_2^*[5F(0;0,5) + 9F(0;1,4) + 5F(0;2,3) + 18F(1;0,4) + 26F(1;1,3)$$
$$+ 28F(2;0,3) + 23F(2;1,2) + 23F(3;0,2) + 9F(4;0,1) - \mathrm{i}F(1,0;0,1,2)] + \text{h.c.}$$

$$j_9^- = \mathrm{i}\psi_1^*\partial_+^9\psi_1 - 201F(1,0;0,7) - 113F(1,0;1,6) - 154F(1,0;2,5) - 224F(1,0;3,4)$$
$$-348F(2,0;0,6) - 265F(2,0;1,5) - 448F(2,0;2,4) - 302F(3,0;0,5) - 521F(3,0;1,4)$$
$$-220F(3,0;2,3) - 111F(2,1;0,5) - 16F(2,1;1,4) + 112F(2,1;2,3) - 146F(4,0;0,4)$$
$$-297F(4,0;1,3) - 48F(3,1;1,3) - 90\mathrm{i}F(2,1,0;0,1,3) + \mathrm{h.c.}$$

$$j_9^+ = \psi_2^*\partial_+^8\psi_1 - \mathrm{i}\psi_2^*[7F(0;0,7) + 20F(0;1,6) + 28F(0;2,5) + 14F(0;3,4) + 177F(1;0,6) - 84F(1;1,5)$$
$$+ 210F(1;2,4) + 107F(2;0,5) + 242F(2;1,4) - 4F(2;2,3) + 55F(3;0,4) + 224F(3;1,3)$$
$$+ 41F(4;0,3) + 32F(4;1,2) + 79F(5;0,2) + 157F(6;01)] - \psi_2^*[-4128F(1,0;0,1,4)$$
$$-614F(1,0;0,2,3) - 4030F(2,0;0,1,3) - 3648F(3,0;0,1,2) + 4838F(2,1;0,1,2)] + \mathrm{h.c.}$$

These classical currents can be shown to be conserved, using only the field eqs. (2) and the anticommutation relations for the $\psi$'s and their derivatives. The explicit form of the currents is, of course, nonunique because of the freedom of adding a curl $\epsilon^{\mu\nu}\partial_\nu f_n$ to $j_n^\mu$, i.e.

$$j_n^- \to j_n^- + \partial^- f_n, \qquad j_n^+ \to j_n^+ - \partial^+ f_n.$$

The terms of $j_n^\pm$ which are products of $2k$ fields are of order $k-1$ in the coupling constant $g$ relative to the bilinear terms. The maximal value of $k$ is a function of $n$, cf. (4a, b). Whereas $j_n^-$ is independent of the mass $m$, all terms of $j_n^+$ are proportional to $m$.

We have benefited from discussions with B. Schroer and R. Seiler.

## References


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