Fachbereich Physik
Freie Universität Berlin
Theoretische Physik

SoSe 02
Set 4
5/15/02

Theorie der Wärme – Statistical Physics
(Prof. E. Frey)

Problem set 4

Problem 4.1 *Paramagnet* (5 pts)
The Hamilton operator for a paramagnet of N particles in an external magnetic field H is

$$\mathcal{H} = -H\sum_{i=1}^{N}\sigma_i\,, \qquad \sigma_i = \pm 1$$

Calculate the partition sum Z and the free energy F of the system. Discuss the mean energy, entropy, and the heat capacity as a function of temperature. Evaluate the mean magnetization $M = -\partial F/\partial H$ and the magnetic susceptibility $\chi = \partial M/\partial H$. Compare the entropy to the result of the microcanonical ensemble.

Problem 4.2 *Harmonic oscillators* (5 pts)
For a system of N identical uncoupled oscillators the energy eigenvalues read

$$E = \sum_{k=1}^{N}\hbar\omega(n_k + \frac{1}{2})\,, \qquad n_k = 0, 1, 2, \ldots$$

Evaluate the partition sum and the free energy. Discuss the mean energy, entropy and the heat capacity. For what temperatures can the quantization of the levels be neglected?

Problem 4.3 *typical states* (4 pts)
Consider a thermodynamic system with discrete energy levels. The canonical ensemble assigns probabilities for the microstates k with corresponding energy E_k according to

$$p_k = Z^{-1}\exp\left(\frac{-E_k}{k_BT}\right)\,, \qquad Z = \sum_k \exp\left(\frac{-E_k}{k_BT}\right)\,.$$

Use the thermodynamic identity $F = -k_BT\ln Z = \langle E\rangle - TS$ to eliminate the partition sum Z in favor of the mean energy $\langle E\rangle$ and the entropy S. Show that the probability for a *typical state* k, i.e. a state with an energy close to the mean one, $|E_k - \langle E\rangle| \leq N\epsilon$, fulfills the bounds

$$e^{-S/k_B}e^{-N\epsilon/k_BT} \leq p_k \leq e^{-S/k_B}e^{N\epsilon/k_BT}\,.$$

Use the extensivity of the heat capacity

$$C_V = \frac{1}{k_BT^2}\langle(E-\langle E\rangle)^2 = \frac{1}{k_BT^2}\sum_k (E_k - \langle E\rangle)^2 p_k$$

to show that in the thermodynamic limit, $N\to\infty$ the *atypical states* have negligible weight, i.e. $\sum' p_k \to 0$, where the prime indicates that the sum is restricted to atypical states. Argue that in the thermodynamic limit all typical microstates are essentially equiprobable, and that the entropy is a measure for the number of these typical states.

Problem 4.4 *free energy* (3 pts)

Consider the probabilites for microstates k

$$p_k(\beta) = Z(\beta)^{-1}\exp(-\beta E_k)\,, \qquad \beta = 1/k_B T$$

of some thermodynamic system as a function of inverse temperature. Use probabilistic arguments to show

$$Z(\beta) = Z(\beta_0)\langle e^{(\beta_0-\beta)E}\rangle_0\,,$$

where $\langle\cdot\rangle_0$ denotes canonical averaging at inverse temperature $\beta_0 = 1/k_B T_0$. Use the definition of the cumulants to show that the corresponding free energies satisfy

$$-\beta F(\beta) = -\beta_0 F(\beta_0) + \sum_{n=1}^{\infty} \kappa_n \frac{(\beta_0-\beta)^n}{n!}$$

where κ_n are the cumulants of the energy with respect to the probability distribution at β_0.

Termine für Übungsgruppen:
Do 12-14 in T3, 1.3.48 (Franosch)
Do 16-18 in T1, 1.3.21 (Parmeggiani, Lattanzi) in englischer Sprache
Fr 12-14 in E2, 1.1.53 (Falcke)
Abgabe: In der Vorlesung vom 22.5.02