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12 Task Theoretical Physics VI - Statistics

12.1 (Bose-Einstein condensation in $d = 1, 2?$)

BEC occurs, if $\mu = \epsilon_0$ (with μ the chemical potential and ϵ_0 the single particle ground state energy). We have $\epsilon_0 = 0$ for an ideal Bose gas without external field. We now want to check, if there is the possibility of BEC for $d = 1, 2$. Therefore we check if the number of particles will converge or diverge in $d = 1, 2$.

Definition of particle number

$$N = \sum_p \frac{1}{e^{\beta(\epsilon_p - \mu)} - 1} = \left(\frac{L}{2\pi\hbar}\right)^d \int \frac{d^d p}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)} - 1}$$

Starting with $d = 1$:

$$N = \frac{L}{2\pi\hbar} \int_0^\infty \frac{dp}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)} - 1}$$

substituting $E = \frac{p^2}{2m} \Leftrightarrow p = \sqrt{2mE}$, $\frac{m}{p} dE = dp$ we get

$$\begin{aligned} N &= \frac{L}{2\pi\hbar} \int_0^\infty \frac{m}{\sqrt{2m}} \frac{dE}{\sqrt{E}} \frac{1}{e^{\beta E - \beta\mu} - 1} \\ &= \frac{\sqrt{m}}{\sqrt{2}} \frac{L}{2\pi\hbar} \int_0^\infty dE \frac{E^{-\frac{1}{2}}}{e^{\beta E - \beta\mu} - 1} \end{aligned}$$

Well, now we have to use a book (at least I have to :-)), since this integral is a pretty hard nut. We start with the definition of the generalized zeta function (from the lecture):

$$g_\nu(z) = \sum_{n=1}^\infty \frac{z^n}{n^\nu}$$

now we will prove the integral form:

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} I_\nu(z)$$

with

$$\begin{aligned} \Gamma(\nu) &= \int_0^\infty t^{\nu-1} e^{-t} dt \\ I_\nu(z) &= \int_0^\infty \frac{x^{\nu-1}}{z^{-1} e^x - 1} \\ &= \int_0^\infty z e^{-x} x^{\nu-1} \cdot \frac{1}{1 - z e^{-x}} \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty z e^{-x} x^{\nu-1} \sum_{n=0}^{\infty} \left(z e^{-x}\right)^n dx \\
&= \sum_{n=1}^{\infty} z^n \int_0^\infty e^{-nx} x^{\nu-1} dx
\end{aligned}
$$

The prove just needs the substitution $x = \frac{t}{n}$ and we get:

$$
\begin{aligned}
I_\nu(z) &= \sum_{n=1}^{\infty} \frac{z^n}{n^{\nu-1}} n \int_0^\infty t^{\nu-1} e^{-t} dt \\
&= \sum_{n=1}^{\infty} \frac{z^n}{n^\nu} \Gamma(\nu) \\
&= g_\nu(z) \Gamma(\nu)
\end{aligned}
$$

therefore:

$$
g_\nu(z) = \frac{1}{\Gamma(\nu)} I_\nu(z)
$$

□

This can now be used on the integral, where we can identify our problem with $I_\nu(z)$.

$$
\begin{aligned}
N &= \frac{\sqrt{m}}{\sqrt{2}} \frac{L}{2\pi\hbar} \beta^{-\frac{1}{2}} \int_0^\infty d\beta E \, \frac{(\beta E)^{\frac{1}{2}-1}}{e^{\beta E} z^{-1} - 1} \\
&= \frac{\sqrt{m}}{\sqrt{2}} \frac{L}{2\pi\hbar} \beta^{-\frac{1}{2}} I_{\frac{1}{2}}(z) \\
&= \frac{\sqrt{m}}{\sqrt{2\beta}} \frac{L}{2\pi\hbar} g_{\frac{1}{2}}(z) \Gamma\left(\frac{1}{2}\right)
\end{aligned}
$$

now inserting $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we get:

$$
N = \frac{L}{2} \frac{\sqrt{2\pi m k_B T}}{h} g_{\frac{1}{2}}(z) = \frac{L}{2\lambda_T} g_{\frac{1}{2}}(z) \to \infty
$$

From the fact that N diverges we gather, that BEC is not possible for $d = 1$, since the number of particle in the ground state will not diverge.

Now $d = 2$:

$$
N = \left(\frac{L}{2\pi\hbar}\right)^2 \int \frac{d^2 p}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)} - 1}
$$

Changing from “cartesian” to “polar” coordinates

$$
\begin{aligned}
N &= \left(\frac{L}{2\pi\hbar}\right)^2 2\pi \int_0^\infty \frac{p \, dp}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)} - 1} \\
&= \frac{L^2 m}{2\pi\hbar^2} \int_0^\infty dE \, \frac{p}{p} \frac{1}{e^{\beta E - \beta\mu} - 1} \\
&= \frac{L^2 m}{2\pi\hbar^2} \int_0^\infty dE \, 1 \cdot \frac{1}{e^{\beta E - \beta\mu} - 1}
\end{aligned}
$$

We want to solve this integral, using

$$\int \frac{dx}{e^x - 1} = \ln\left(1 - e^{-x}\right) \Leftrightarrow \frac{\partial}{\partial x} \ln\left(1 - e^{-x}\right) = \frac{e^{-x}}{1 - e^{-x}} = \frac{1}{e^x - 1}$$

this leads to

$$\begin{aligned} N &= \frac{L^2 m}{2\pi\hbar^2} \left[\ln\left(1 - e^{-\beta E - \beta\mu}\right)\right]_0^\infty \\ &= \frac{L^2 m}{2\pi\hbar^2} \left[\ln(1) - \ln\left(1 - e^{-\beta\mu}\right)\right] \\ &= -\frac{L^2 m}{2\pi\hbar^2} \ln\left(1 - e^{-\beta\mu}\right) \end{aligned}$$

with $\mu \to 0$ the argument of the logarithm will go to 0, but while $\lim_{\epsilon\to 0} - \ln(\epsilon) \to \infty$, N will not converge for $d = 2$. This means, the ground state particle number is small compared to the total number of particles, therefore there is no BEC for $d = 2$.

Alternatively with the help of the generalized zeta function we can identify

$$\begin{aligned} N &= \frac{L^2 m}{2\pi\hbar^2} \beta^{-1} \int_0^\infty d\beta E \, 1 \cdot \frac{(\beta E)^{1-1}}{e^{\beta E} z^{-1} - 1} \\ &= \frac{L^2 m}{2\pi\hbar^2} \beta^{-1} I_1(z) \\ &= \frac{2\pi m k_B T}{h^2} L^2 g_1(z) \Gamma(1) \end{aligned}$$

which will not converge:

$$N = \frac{L^2}{\lambda_T^2} g_1(z) \to \infty$$

12.2 (Bose-Einstein condensation in a harmonic potential)

(a)

The single particle Hamiltonian is given with

$$\hat{H} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2\right) + \frac{1}{2} m \left(\omega_x^2 \hat{x}^2 + \omega_y^2 \hat{y}^2 + \omega_z^2 \hat{z}^2\right)$$

leading to the well known energy value:

$$\epsilon = \sum_{i=1}^{3} E_i = \sum_{i=1}^{3} \hbar\omega_i \left(n_i + \frac{1}{2}\right)$$

We first consider the possible energy states, while we have the condition $E_i \geq 0$ (with $n_i \geq 0$) we can only have values in one octant. Furthermore we define each energy state to have a rectangular energy unit cell surrounding it with the volume $V_\epsilon = \hbar\omega_1\hbar\omega_2\hbar\omega_3 = (\hbar\bar{\omega})^3$. If we fix ϵ we will receive a plane in this octant (this means an octahedron if we would look at all octants), intercepting $(\epsilon, 0, 0)$, $(0, \epsilon, 0)$ and $(0, 0, \epsilon)$, a second plane with $\epsilon + \Delta\epsilon$ will now be parallel to the first one but with the intercepts $(\epsilon + \Delta\epsilon, 0, 0)$, $(0, \epsilon + \Delta\epsilon, 0)$ and $(0, 0, \epsilon + \Delta\epsilon)$. Therefore we can interpret it as the difference of the density of states, while considering only the first case of a fixed ϵ would give us $D(\epsilon)$ while the second plane will lead to $D(\epsilon + \Delta\epsilon)$. While $\epsilon \gg \hbar\bar{\omega}$ the ground state energy can be neglected. We also get a quasi continous density while before we only had points in space, but now we can consider a real density, because of the huge number of possible states $N \gg 1$. Now we can find the density of states using $\Omega = \frac{V}{8}$ and the V for an octahedron $V = \frac{\sqrt{2}}{3}a^3$. With a the length of one octahedron side, while we get $a = \sqrt{2}\epsilon$ if we think of our used model, while we can use the intercepts and the connecting line of two intercepts is a. Now using the definition of the density of states we get:

$$D(\epsilon) = \frac{1}{V_\epsilon}\frac{d\Omega}{d\epsilon} = \frac{1}{2^3(\hbar\bar{\omega})^3}\frac{d}{d\epsilon}\frac{\sqrt{2}}{3}\left(\sqrt{2}\epsilon\right)^3 = \frac{2^2 3}{2^3 3(\hbar\bar{\omega})^3}\epsilon^2 = \frac{\epsilon^2}{2(\hbar\bar{\omega})^3}$$

(b)

We consider N atoms in the trap. This is given by

$$N = \int_0^\infty d\epsilon D(\epsilon)\, n_{BE}(\epsilon)$$

with

$$n_{BE}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} - 1}$$

we can insert $D(\epsilon)$ to get

$$\begin{aligned}
N &= \frac{1}{2(\hbar\bar{\omega})^3}\int_0^\infty d\epsilon\epsilon^2 n_{BE}(\epsilon) \\
&= \frac{1}{2(\beta\hbar\bar{\omega})^3}\int_0^\infty d\beta\epsilon\frac{(\beta\epsilon)^{3-1}}{e^{\beta\epsilon}z^{-1} - 1} \\
&= \frac{1}{2(\beta\hbar\bar{\omega})^3} g_3(z)\,\Gamma(3) \\
&= \frac{1}{2(\beta\hbar\bar{\omega})^3} g_3(z)\, 2 \\
&= \frac{g_3(z)}{(\beta\hbar\bar{\omega})^3}
\end{aligned}$$

this means:

$$N = \frac{g_3\left(e^{\beta\mu}\right)}{(\beta\hbar\bar{\omega})^3}$$

just an implicit form of μ. But rearrangement seems not to be possible at the moment. The only thing we can do is an approximation, using

$$g_3(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^3}$$

and knowing that for bigger n this will be small because n^3 in the denominator will become huge, while z in the numerator is about 1, we just simply take the first order:

$$g_3(z) \approx z = e^{\beta\mu}$$

Therefore we get the easy expression:

$$\mu \approx \frac{1}{\beta}\left[\ln N + 3\ln(\beta\hbar\bar{\omega})^3\right]$$

We are looking for T_c the Bose-Einstein-condensation temperature, we can set $\mu = 0$ which means $z = 1$ and we therefore get:

$$N = \frac{k_B^3 T_c^3 g_3(1)}{(\hbar\bar{\omega})^3}$$

rearranging for T_c and using $g_\nu(1) = \zeta(\nu)$

$$T_c = \frac{\hbar\bar{\omega}}{k_B}\left(\frac{N}{g_3(1)}\right)^{\frac{1}{3}} = \frac{\hbar\bar{\omega}}{k_B}\left(\frac{N}{\zeta(3)}\right)^{\frac{1}{3}} \tag{1}$$

After we found the critical temperature, we now want to compare the characteristic size of the trap to the thermal de Broglie wave to give a physical interpretation. It seems quite likely (also we don't have that for **d)**) to approximate by a symmetric trap (we only need the approximate order of the trap size for comparison), while the size seems to be determined by the harmonic potential. We therefore want to find the expectation value for $\vec{r}$ which we can get using the ground state energy, with $\vec{p} = 0$ leading to

$$\hat{H} = 0 + \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right) \approx \frac{m}{2}\bar{\omega}^2\vec{r}^2$$

For further approximation we use the ground state energy with $\omega_x = \omega_y = \omega_z = \bar{\omega}$ meaning $E_0 = \frac{1}{2}\hbar(\omega_x + \omega_y + \omega_z) \approx \frac{3}{2}\hbar\bar{\omega}$. Putting things together we get:

$$\frac{m}{2}\bar{\omega}^2\left\langle\vec{r}^2\right\rangle \approx \left\langle\hat{H}\right\rangle = \langle E_0\rangle \approx \frac{3}{2}\hbar\bar{\omega}$$

now we can rearrange this for r:

$$r = \sqrt{\frac{3\hbar}{m\bar{\omega}}}$$

Now the definition of thermal de Broglie wave length is

$$\lambda_{T_c} = \frac{h}{\sqrt{2\pi m k_B T_c}}$$

with equation (1) this is:

$$\lambda_{T_c}=\sqrt{\frac{h}{m\bar{\omega}}}\left(\frac{\zeta(3)}{N}\right)^{\frac{1}{6}}=\sqrt{\frac{1}{6\pi}}\sqrt{\frac{3\hbar}{m\bar{\omega}}}\left(\frac{\zeta(3)}{N}\right)^{\frac{1}{6}}=r\cdot\sqrt{\frac{1}{6\pi}}\left(\frac{\zeta(3)}{N}\right)^{\frac{1}{6}}$$

The comparison now only yields the factor of

$$\frac{\lambda_{T_c}}{r}=\sqrt{\frac{1}{6\pi}}\left(\frac{\zeta(3)}{N}\right)^{\frac{1}{6}}$$

for the thermal de Broglie wavelength compared to the characteristic size of the trap. Using $N=10^7$ like given in **d)** we get a factor of

$$\frac{\lambda_{T_c}}{r}\approx 0.016=1.6\cdot 10^2$$

This means the de Broglie wave length is about 2 orders smaller than the characteristic size of the trap. This will force the particles together to a certain amount, since there are 10^7 particles in there, increasing the density of particles in the potential.

To get an accurate value of T_c we first have to consider, what the dependencies for T_c are. Looking at equation (1) we see some constants, but also the dependence of $\bar{\omega}$ meaning dependence on $\omega_x,\omega_y,\omega_z$ and on N. Our approximation is only valid for a high N since otherwise, we won't have a continous density. Also we considered a symmetric case for the comparison between trap size and thermal de Broglie wavelength, which in experimental setups is quite hard to establish, but while in task **d)** we get an example, where we don't have a symmetric potential it is likely that the symmetry doesn't matter for the T_c too much, there is no reason why it should anyway. While N also influence the thermal de Broglie wavelength it seems likely, that the size of the trap should matter (at least in comparison to the thermal de Broglie wavelength), this brings $\omega_x,\omega_y,\omega_z$ back in game, while if we had a huge trap it doesn't seem likely, that all particles would be in the same state (at least those one in New York and Berlin will never form a BEC).

(c)

To find an expression for the condensate density for $T<T_c$ we can first split our particle number like we did in the lecture.

$$N=\sum_p n(\varepsilon_p)=n(0)+\sum_{p\neq 0}n(\varepsilon_p)$$

We now have to go to the integral $\sum_p\rightarrow\int d^3p$. We get:

$$\begin{aligned}N&=n(0)+\sum_{p\neq 0}n(\varepsilon_p)\\&=n(0)+\frac{1}{2(\hbar\bar{\omega})^3}\int_{E_0}^{\infty}d\epsilon\epsilon^2 n_{BE}(\epsilon)\end{aligned}$$

substituting $x=\epsilon-E_0$ we get:

$$N = n(0) + \frac{1}{2(\hbar\bar{\omega})^3} \int_0^\infty dx \frac{x^2 - 2xE_0 + E_0^2}{e^{\beta x} z^{-1} - 1}$$

(with $z^{-1} = e^{\beta\mu - \beta E_0}$) which can be split into 3 integrals

$$N = n(0) + \frac{1}{2(\hbar\bar{\omega})^3} \left(\frac{1}{\beta^3} \int_0^\infty d\beta x \frac{(\beta x)^{3-1}}{e^{\beta x} z^{-1} - 1} - \frac{2E_0}{\beta^2} \int_0^\infty d\beta x \frac{(\beta x)^{2-1}}{e^{\beta x} z^{-1} - 1} + \frac{E_0^2}{\beta} \int_0^\infty d\beta x \frac{(\beta x)^{1-1}}{e^{\beta x} z^{-1} - 1} \right)$$

rewriting the integrals we get:

$$N = n(0) + \frac{1}{2(\hbar\bar{\omega})^3} \left(\frac{1}{\beta^3} g_3(z) \Gamma(3) - \frac{2E_0}{\beta^2} g_2(z) \Gamma(2) + \frac{E_0^2}{\beta} g_1(z) \Gamma(1) \right)$$

now inserting the Γ functions leading to

$$N = n(0) + \frac{1}{2(\hbar\bar{\omega})^3} \left(\frac{2}{\beta^3} g_3(z) - \frac{2E_0}{\beta^2} g_2(z) + \frac{E_0^2}{\beta} g_1(z) \right)$$

now using $E_0 \to 0$ (well doesn't seem to be all right, since $g_1(z)$ will lead to complex infinity, therefore we want to get rid of this term) we can simplify this to

$$N \approx n(0) + \frac{g_3(z)}{(\beta\hbar\bar{\omega})^3}$$

inserting $N = \frac{k_B^3 T_c^3 g_3(1)}{(\hbar\bar{\omega})^3} \Leftrightarrow (\hbar\bar{\omega})^3 = \frac{k_B^3 T_c^3 g_3(1)}{N}$ we get

$$N = n(0) + N \frac{g_3(z) k_B^3 T^3}{k_B^3 T_c^3 g_3(1)}$$

rearranged

$$n(0) = N \left[1 - \frac{g_3(z) T^3}{g_3(1) T_c^3} \right]$$

and with $z = 1$ we have

$$n(0) = N \left[1 - \left(\frac{T}{T_c} \right)^3 \right]$$

In the lecture we found the exponent of $\frac{3}{2}$ for the case without external potential, therefore this can be interpreted as an effect of the potential. The potential seems to increase the number of BEC particles faster than in the free state. This seems likely, while they are forced together by the potential.

(d)

Using

$$T_c = \frac{h\frac{\bar{\omega}}{2\pi}}{k_B}\left(\frac{N}{\zeta(3)}\right)^{\frac{1}{3}}$$

and inserting $\frac{1}{2\pi}(\omega_x, \omega_y, \omega_z) = (250, 670, 7)$, $N = 10^7$ and $\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \ldots = 1.202\ldots$ we get the critical temperature

$$T_c = 1.03 \cdot 10^{-6}\,\mathrm{K}$$

which seems quite small. Since we know that for Helium we can get temperatures of about $1 - 2\,\mathrm{K}$ and observe BEC.

12.3 (Bose-Einstein condensation in a gravitational field)

Energy eigenvalues in homogeneous gravitational field, within a $V = L^3$ box, consisting of potential and kinetic part

$$E = E_{kin} + E_{pot} = \frac{p^2}{2m} + mgh$$

each particle got mass m and inner degree of freedom ω

(a)

Grand partition function

$$\ln Z_G = -\omega \ln\left(1 - e^{-\beta mgh} z\right) + \omega \frac{V}{\lambda_T^3} \sum_{N=0}^{\infty} \frac{z^{N+1}}{(N+1)^{\frac{7}{2}}} \frac{1 - e^{-\beta mgL(N+1)}}{\beta mgL} \qquad (2)$$

We will just do the pretty basic stuff and then skip the rest. Therefore we start with knowledge from the lecture, where we showed that

$$Z_G = \Pi_i \left[1 - e^{-\beta(\varepsilon_i - \mu)}\right]^{-\omega}$$

Just using the ln on that leads us to

$$\ln Z_G = -\omega \ln \Pi_i \left[1 - e^{-\beta(\varepsilon_i - \mu)}\right]$$

while we got the given form of the result it seems worthy to seperate the ground state from the rest of the energy states, while the ground state should be independent of the kinetic term, we just get mgh and therefore:

$$\begin{aligned}\ln Z_G &= -\omega \sum_i \ln\left[1 - e^{-\beta(\varepsilon_i - \mu)}\right] \\ &= -\omega \ln\left(1 - e^{-\beta mgh} e^{\beta\mu}\right) - \omega \sum_{i=1}^{N} \ln\left[1 - e^{-\beta(\varepsilon_i - \mu)}\right]\end{aligned}$$

but this is already first part of the task with $\xi = e^{-\beta mgh}z$ and $e^{\beta\mu} = z$ we get:

$$\ln Z_G = -\omega \ln(1-\xi) - \omega \sum_{i=1}^{N} \ln\left[1 - e^{-\beta(\varepsilon_i - \mu)}\right]$$

It now remains to show that

$$-\omega \sum_{i=1}^{N} \ln\left[1 - e^{-\beta(\varepsilon_i - \mu)}\right] = \omega \frac{V}{\lambda_T^3} \sum_{N=0}^{\infty} \frac{z^{N+1}}{(N+1)^{\frac{7}{2}}} \frac{1 - e^{-\beta mgL(N+1)}}{\beta mgL}$$

which we

skipped

But for completeness the total result should be:

$$\ln Z_G = -\omega \ln(1-\xi) + \omega \frac{V}{\lambda_T^3} \sum_{N=0}^{\infty} \frac{z^{N+1}}{(N+1)^{\frac{7}{2}}} \frac{1 - e^{-\beta mgL(N+1)}}{\beta mgL}$$

(b)

Definition particle number:

$$\begin{aligned} N &= \frac{\partial \Phi}{\partial \mu} \\ &= \frac{1}{\beta} \frac{\partial \ln Z_G}{\partial \mu} \end{aligned}$$

inserting $\ln Z_G$ from equation (2) we can see, that the only μ dependence is in $z = e^{\beta\mu}$ the fugacity

$$N = \frac{1}{\beta} \frac{\partial}{\partial \mu} \left(-\omega \ln\left(1 - e^{-\beta mgh} z\right) + \omega \frac{V}{\lambda_T^3} \sum_{N=0}^{\infty} \frac{z^{N+1}}{(N+1)^{\frac{7}{2}}} \frac{1 - e^{-\beta mgL(N+1)}}{\beta mgL} \right)$$

therefore we rewrite the differentiation:

$$\frac{\partial}{\partial \mu} = \frac{\partial z}{\partial \mu} \frac{\partial}{\partial z} = \beta z \frac{\partial}{\partial z}$$

This leads to:

$$\begin{aligned} N &= z \frac{\partial}{\partial z} \left(-\omega \ln\left(1 - e^{-\beta mgh} z\right) + \omega \frac{V}{\lambda_T^3} \sum_{N=0}^{\infty} \frac{z^{N+1}}{(N+1)^{\frac{7}{2}}} \frac{1 - e^{-\beta mgL(N+1)}}{\beta mgL} \right) \\ &= -z\omega \frac{-e^{-\beta mgh}}{1 - e^{-\beta mgh} z} + z\omega \frac{V}{\lambda_T^3} \sum_{N=0}^{\infty} \frac{(N+1)\, z^{N+1}}{(N+1)^{\frac{7}{2}}} \frac{1 - e^{-\beta mgL(N+1)}}{\beta mgL} \\ &= \frac{\omega}{e^{\beta mgh} z - 1} + \frac{\omega V}{\beta mgL \lambda_T^3} \sum_{N=0}^{\infty} \frac{z^{N+1}}{(N+1)^{\frac{5}{2}}} \left(1 - e^{-\beta mgL(N+1)}\right) \end{aligned}$$

We can now move the index of the sum to $N = 1$ and identify it with the generalized zeta function:

$$g_\nu(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\nu}$$

and we got:

$$N = \frac{\omega}{e^{\beta mgh} z - 1} + \frac{\omega V}{\beta mgL\lambda_T^3} \sum_{N=1}^{\infty} \left(\frac{z^N}{N^{\frac{5}{2}}} - \frac{\left(ze^{-\beta mgL}\right)^N}{N^{\frac{5}{2}}} \right)$$

now using small T, meaning β will become big, the first term will cancel to the second, since $e^{-\beta} \to 0$. It only remains:

$$N = \frac{\omega V}{\beta mgL\lambda_T^3} \sum_{N=1}^{\infty} \left(\frac{z^N}{N^{\frac{5}{2}}} - \frac{\left(ze^{-\beta mgL}\right)^N}{N^{\frac{5}{2}}} \right)$$

inserting gerneralized zeta function definition and definition of fugacity $z = e^{\beta\mu}$ we get

$$\begin{aligned} N &= \frac{\omega V}{\beta mgL\lambda_T^3} \left(g_{\frac{5}{2}}(z) - g_{\frac{5}{2}}\left(ze^{-\beta mgL}\right) \right) \\ &= \frac{\omega V}{\beta mgL\lambda_T^3} \left(g_{\frac{5}{2}}\left(e^{\mu\beta}\right) - g_{\frac{5}{2}}\left(e^{\beta(\mu - mgL)}\right) \right) \end{aligned}$$

with the approximation suggested in the task this becomes:

$$\begin{aligned} N &\approx \frac{\omega V}{\beta mgL\lambda_T^3} \left(g_{\frac{5}{2}}(1) - \mu\beta g_{\frac{3}{2}}(1) + \frac{4}{3}\sqrt{\pi}(\mu\beta)^{\frac{3}{2}} + \mathcal{O}\left((\mu\beta)^2\right) \right) \\ &- \frac{\omega V}{\beta mgL\lambda_T^3} \left(g_{\frac{5}{2}}(1) - \beta(\mu - mgL) g_{\frac{3}{2}}(1) + \frac{4}{3}\sqrt{\pi}(\beta(\mu - mgL))^{\frac{3}{2}} + \mathcal{O}\left((\beta(\mu - mgL))^2\right) \right) \\ &= \frac{\omega V}{\beta mgL\lambda_T^3} \left(\beta mgL g_{\frac{3}{2}}(1) + \frac{4}{3}\sqrt{\pi}\left[(\mu\beta)^{\frac{3}{2}} - (\beta(\mu - mgL))^{\frac{3}{2}}\right] + \mathcal{O}\left((\mu\beta)^2\right) \right) \\ &\approx \frac{\omega V}{\lambda_T^3} g_{\frac{3}{2}}(1) + \frac{4\omega V\sqrt{\pi}}{3\beta mgL\lambda_T^3} \left[(\mu\beta)^{\frac{3}{2}} - (\beta(\mu - mgL))^{\frac{3}{2}}\right] \end{aligned}$$

Now Bose-Einstein condensation corresponds to $z = 1$ this means $\beta\mu = 0$ this leads to:

$$N = \frac{\omega V}{\lambda_T^3} \left[g_{\frac{3}{2}}(1) - \frac{4}{3}(\pi\beta mgL)^{\frac{1}{2}} \right]$$

and dividing through V this is

$$n = \frac{N}{V} = \frac{\omega}{\lambda_T^3} \left[g_{\frac{3}{2}}(1) - \frac{4}{3}(\pi\beta mgL)^{\frac{1}{2}} \right]$$

(c)

We already found the Term N in **b)** which depends on T and we found it for $z = 1$ which corresponds to BEC, which only occurs for $T \leq T_c$. If we consider the case $T = T_c$ we can get an equation for T_c from this

$$n = \frac{\omega}{\lambda_{T_c}^3}\left[g_{\frac{3}{2}}(1) - \frac{4}{3}\left(\frac{\pi}{k_B T_c} mgL\right)^{\frac{1}{2}}\right]$$

with

$$\lambda_{T_c}^3 = \left(\frac{h}{\sqrt{2\pi m k_B T_c}}\right)^3$$

and for constant n, which we have while the total particle number N is conserved and we the constraint of a box with solid borders means V is also constant, we have from the lecture for $g = 0$

$$n\lambda_{T_c}^3 = g_{\frac{3}{2}}(1)$$

which verifies our equation for $g = 0$. We can now insert all we got and get:

$$n\left(\frac{h}{\sqrt{2\pi m k_B}}\right)^3 T_c^{-\frac{3}{2}} = \omega g_{\frac{3}{2}}(1) - \frac{4}{3}\omega\left(\frac{\pi}{k_B} mgL\right)^{\frac{1}{2}} T_c^{-\frac{1}{2}}$$

rewriting this for T_c isn't easy, just the first step

$$\frac{n}{\omega}\left(\frac{h}{\sqrt{2\pi m k_B}}\right)^3 = T_c^{\frac{3}{2}} g_{\frac{3}{2}}(1) - \frac{4}{3}\left(\frac{\pi}{k_B} mgL\right)^{\frac{1}{2}} T_c^{\frac{1}{2}}$$

now we want to simplify this a little bit using some constants

$$\begin{aligned} a &= \frac{n}{\omega}\left(\frac{h}{\sqrt{2\pi m k_B}}\right)^3 \\ b &= \frac{4}{3}\left(\frac{\pi}{k_B} mgL\right)^{\frac{1}{2}} \\ c &= g_{\frac{3}{2}}(1) = \zeta\left(\frac{3}{2}\right) \end{aligned}$$

we therefore get an equation of the form

$$\begin{aligned} cT_c^{\frac{3}{2}} - bT_c^{\frac{1}{2}} - a &= 0 \\ T_c^{\frac{3}{2}} - \frac{b}{c} T_c^{\frac{1}{2}} - \frac{a}{c} &= 0 \end{aligned}$$

We now have to solve it for T_c, mathematica gives a solution, but it seems pretty lengthy, therefore we guess, there must have been something wrong. We give the easier solution for $g = 0$ anyway:

$$T_c^0 = \left(\frac{a}{c}\right)^{\frac{2}{3}} = \left(\frac{\frac{n}{\omega}\left(\frac{h}{\sqrt{2\pi m k_B}}\right)^3}{g_{\frac{3}{2}}(1)}\right)^{\frac{2}{3}} = \frac{h^2}{2\pi m k_B}\left(\frac{n}{\omega\zeta\left(\frac{3}{2}\right)}\right)^{\frac{2}{3}}$$

Now we can decide, whether T_c^0 or T_c will be larger, since we got the equation:

$$\frac{n}{\omega}\left(\frac{h}{\sqrt{2\pi m k_B}}\right)^3 = T_c^{\frac{3}{2}} g_{\frac{3}{2}}(1) - \frac{4}{3}\left(\frac{\pi}{k_B} mgL\right)^{\frac{1}{2}} T_c^{\frac{1}{2}}$$

and the left side is constant, while for $g = 0$ the right side will be larger than for $g \neq 0$. Therefore $T_c < T_c^0$. Going further now to find T_c in an approximation

$$\frac{n}{\omega}\left(\frac{h}{\sqrt{2\pi m k_B T_c}}\right)^3 T_c^{\frac{3}{2}} = T_c^{\frac{3}{2}} g_{\frac{3}{2}}(1) - \frac{4}{3}\left(\frac{\pi}{k_B} mgL\right)^{\frac{1}{2}} T_c^{\frac{1}{2}}$$

Inserting $n\lambda_{T_c}^3 = g_{\frac{3}{2}}(1)$ which is valid for $g = 0$ we get:

$$g_{\frac{3}{2}}(1)\left(\frac{1}{\omega} - 1\right) T_c = -\frac{4}{3}\left(\frac{\pi}{k_B} mgL\right)^{\frac{1}{2}}$$

which can be seen as an approximation for the case of $g \neq 0$:

$$T_c = \frac{4}{3}\omega\frac{\left(\frac{\pi}{k_B} mgL\right)^{\frac{1}{2}}}{\zeta\left(\frac{3}{2}\right)(\omega - 1)}$$