Heiko Dumlich November 13, 2007

4 Task Theoretical Physics VI - Statistics

4.1 (Spin lattice)

At each site of a isolated lattice an unpaired electron with spin 1/2 is localized. External magnetic field $\vec{B} = B\vec{e}_z$, spins parallel or anti-parallel to magnetic field $\left(s_{z,i} = \pm\frac{1}{2}\right)$. Energy of state r given by:

$$E_r(B) = -2\mu_B B \sum_{i=1}^{N} s_{z,i}$$

(a)

n spins parallel to magnetic field $\Rightarrow N-n$ spins anti-parallel. Energy in depandance of n:

$$\begin{aligned} E_n(B) &= -2\mu_B B\left(\frac{1}{2}\sum_{i=1}^{n} 1 - \frac{1}{2}\sum_{i=n}^{N} 1\right) \\ &= -2\mu_B B\left(\frac{1}{2}\sum_{i=1}^{n} 1 - \frac{1}{2}\sum_{i=1}^{N-n} 1\right) \\ &= -2\mu_B B\left(n - \frac{N}{2}\right) \end{aligned}$$

Number of states given through binomial distribution:

$$\Omega_n(E_n) = \frac{N!}{n!\,(N-n)!}$$

with:

$$n(E_n) = \frac{N}{2}\left(1 - \frac{E_n(B)}{N\mu_B B}\right)$$

this leads to:

$$\Omega_n(E_n) = \frac{N!}{\left(\frac{N}{2}\left(1 - \frac{E_n(B)}{N\mu_B B}\right)\right)!\left(\frac{N}{2}\left(1 + \frac{E_n(B)}{N\mu_B B}\right)\right)!}$$

(b)

Applying Stirling's formula to $\Omega_n(E_n)$, with $N-n \gg 1 \Rightarrow N \gg n$ and $n \gg 1$:

$$\Omega_n(E_n) = \frac{\left(\frac{N}{e}\right)^N}{\left(\frac{n}{e}\right)^n\left(\frac{N-n}{e}\right)^{N-n}} = \frac{N^N e^n e^{-N} e^{N-n}}{n^n (N-n)^{N-n}} = \frac{N^N}{n^n (N-n)^{N-n}}$$

Logarithmus naturalis:

$$\ln \Omega_n (E_n) = N \ln N - n \ln n - (N-n) \ln (N-n)$$

inserting $n(E_n)$:

$$\begin{aligned}
\ln \Omega_n (E_n) &= N \ln N - \frac{N}{2}\left(1 - \frac{E_n(B)}{N\mu_B B}\right) \ln \frac{N}{2}\left(1 - \frac{E_n(B)}{N\mu_B B}\right) \\
&- \left(N - \frac{N}{2}\left(1 - \frac{E_n(B)}{N\mu_B B}\right)\right) \ln \left(N - \frac{N}{2}\left(1 - \frac{E_n(B)}{N\mu_B B}\right)\right) \\
&= N \ln N - \frac{N}{2}\left(1 - \frac{E}{N\mu_B B}\right) \ln \left[N\left(\frac{1}{2} - \frac{E}{2N\mu_B B}\right)\right] \\
&- \frac{N}{2}\left(1 + \frac{E}{N\mu_B B}\right) \ln \left[N\left(\frac{1}{2} + \frac{E}{2\mu_B B}\right)\right] \\
&= N \ln N - \frac{N}{2}\left(1 - \frac{E}{N\mu_B B}\right) \left\{\ln N + \ln \left(\frac{1}{2} - \frac{E}{2N\mu_B B}\right)\right\} \\
&- \frac{N}{2}\left(1 + \frac{E}{N\mu_B B}\right) \left\{\ln N + \ln \left(\frac{1}{2} + \frac{E}{2\mu_B B}\right)\right\} \\
&= N \ln N - \left(\frac{N}{2} - \frac{E}{2\mu_B B}\right) \ln N - \frac{N}{2}\left(1 - \frac{E}{N\mu_B B}\right) \ln \left(\frac{1}{2} - \frac{E}{2N\mu_B B}\right) \\
&+ - \left(\frac{N}{2} + \frac{E}{2\mu_B B}\right) \ln N - \frac{N}{2}\left(1 + \frac{E}{N\mu_B B}\right) \ln \left(\frac{1}{2} + \frac{E}{2\mu_B B}\right) \\
&= -\frac{N}{2}\left(1 - \frac{E}{N\mu_B B}\right) \ln \left(\frac{1}{2} - \frac{E}{2N\mu_B B}\right) - \frac{N}{2}\left(1 + \frac{E}{N\mu_B B}\right) \ln \left(\frac{1}{2} + \frac{E}{2\mu_B B}\right) \\
&+ \left(N - \frac{N}{2} + \frac{E}{2\mu_B B} - \frac{N}{2} - \frac{E}{2\mu_B B}\right) \ln N \\
&= -\frac{N}{2}\left(1 - \frac{E}{N\mu_B B}\right) \ln \left(\frac{1}{2} - \frac{E}{2N\mu_B B}\right) - \frac{N}{2}\left(1 + \frac{E}{N\mu_B B}\right) \ln \left(\frac{1}{2} + \frac{E}{2\mu_B B}\right)
\end{aligned}$$

□

4.2 (Mixture of ideal gases)

Mixture of ideal gases with N_i particles of the kind i ($i = 1, \ldots, m$) in a vessel of volume V.

Definition of entropy:

$$S = -k_B \ln W(E)$$

with $W(E) = \frac{\Omega(E)}{N!h^{3N}}$, for indistingushable particles with $c_N = \frac{1}{N!h^{3N}}$.
For the total system we can write:

$$\Omega(N_1, \ldots, N_m) = \frac{N!}{\Pi_{i=1}^{m} N_i!} \Pi_{i=1}^{m} \Omega_i^{N_i}$$

Inserting this into definition of the Entropie:

$$
\begin{aligned}
S &= -k_B \ln \frac{\Omega\left(N_1, \ldots, N_m\right)}{N! h^{3N}} \\
&= -k_B \ln \frac{\frac{N!}{\Pi_{i=1}^m N_i!} \Pi_{i=1}^m \Omega_i^{N_i}}{N! h^{3N}} \\
&= -k_B \left(\ln \frac{N!}{\Pi_{i=1}^m N_i!} + \sum_{i=1}^m N_i \ln \Omega_i - \ln N! h^{3N} \right)
\end{aligned}
$$

Applying Stirling formula $N!/\Pi_{i=1}^m N_i! = \Pi_{i=1}^m \left(\frac{N}{e}\right)^N \left(\frac{e}{N_i}\right)^{N_i} \approx \Pi_{i=1}^m \left(\frac{N}{N_i}\right)^{N_i}$:

$$
\begin{aligned}
S_{\text{ensemble}} &= -k_B \left(\ln \Pi_{i=1}^m \left(\frac{N}{N_i}\right)^{N_i} + \sum_{i=1}^m N_i \ln \Omega_i - \ln N! h^{3N} \right) \\
&= -k_B \left(\ln \Pi_{i=1}^m \left(\frac{N}{N_i}\right)^{N_i} + \sum_{i=1}^m N_i \ln \frac{\Omega_i}{N! h^{3N}} \right) \\
&= -k_B \sum_{i=1}^m N_i \ln \frac{N}{N_i} - k_B \sum_{i=1}^m N_i \ln \frac{\Omega_i}{N! h^{3N}} \\
&= -k_B \sum_{i=1}^m N_i \ln \frac{N}{N_i} + S \\
&= S_{\text{Mix}} + S
\end{aligned}
$$

Entropy of an ideal gas (Sackur-Tetrode-equation):

$$
S\left(E, V, N\right) = k_B N \left[\ln\left(\frac{V}{N}\right) + \frac{3}{2} \ln\left(\frac{4\pi m E}{3 N h^2}\right) + \frac{5}{2} \right]
$$

leads to:

$$
\begin{aligned}
S\left(E, V, N_1, N_2, \ldots N_m\right) &= \sum_{i=1}^m S_i + S_{\text{Mix}} \\
&== \sum_{i=1}^m k_B N_i \left[\ln\left(\frac{V}{N_i}\right) + \frac{3}{2} \ln\left(\frac{4\pi m_i E_i}{3 N_i h^2}\right) + \frac{5}{2} \right] - k_B \sum_{i=1}^m N_i \ln \frac{N}{N_i} \\
&= \sum_{i=1}^m k_B N_i \left(\ln \frac{V}{N_i} - \ln \frac{N}{N_i} + \frac{3}{2} \ln\left(\frac{4\pi m_i E_i}{3 N_i h^2}\right) + \frac{5}{2} \right) \\
&= \sum_{i=1}^m k_B N_i \left(\ln\left(\frac{V}{N_i} \frac{N_i}{N}\right) + \frac{3}{2} \ln\left(\frac{4\pi m_i E_i}{3 N_i h^2}\right) + \frac{5}{2} \right) \\
&= \sum_{i=1}^m k_B N_i \left(\ln \frac{V}{N} + \frac{3}{2} \ln\left(\frac{4\pi m_i E_i}{3 N_i h^2}\right) + \frac{5}{2} \right) \\
&= k_B N \ln \frac{V}{N} + \sum_{i=1}^m k_B N_i \frac{3}{2} \ln\left(\frac{4\pi m_i E_i}{3 N_i h^2}\right) + \frac{5}{2} k_B N
\end{aligned}
$$

Definition of pressure:

$$p = T \frac{\partial S}{\partial V}\Big|_{E,N}$$

inserting S:

$$p = k_B T \frac{1}{V} N = \frac{1}{\beta} \frac{N}{V} = \frac{\rho}{\beta} \Leftrightarrow pV = N k_B T$$

4.3 (Ideal gas in a centrifuge)

Radius$= R$, height$= L$, centrifuge frequency$= \omega$ and mass of particle$= m$. The z-axis is parallel to the axis of the centrifuge. One particle Hamiltonian:

$$H = \frac{p^2}{2m} - \omega \left(x p_y - y p_x \right)$$

(a)

To show:

$$Z_1 = \left(\frac{2\pi m}{h^2 \beta} \right)^{3/2} \frac{2\pi L}{m \beta \omega^2} \left(e^{m\beta\omega^2 R^2/2} - 1 \right)$$

partition function for one particle of the gas.
Definition of partition function:

$$Z_N = c_N \int d^{3N} q \, d^{3N} p \, e^{-\beta H(\vec{q}_N, \vec{p}_N)}$$

with $c_N = \frac{1}{N! h^{3N}}$ for identical particles.
Inserting Hamiltonian:

$$Z_1 = \frac{1}{1! h^3} \int d^3 q \, d^3 p \, e^{-\beta \frac{p^2}{2m} + \beta\omega (x p_y - y p_x)}$$

Integrating over momentum:

$$Z_1 = \frac{1}{1! h^3} \int d^3 q \int_{-\infty}^{\infty} dp_x \, e^{-\frac{\beta}{2m} p_x^2 - \beta\omega y p_x} \int_{-\infty}^{\infty} dp_y \, e^{-\frac{\beta}{2m} p_y^2 + \beta\omega x p_y} \int_{-\infty}^{\infty} dp_z \, e^{-\frac{\beta}{2m} p_z^2}$$

with:

$$\begin{aligned}
\int_{-\infty}^{\infty} dp_z \, e^{-\frac{\beta}{2m} p_z^2} &= \sqrt{\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \exp\left(-\frac{\beta}{2m} (x^2 + y^2) \right)} \\
&= \sqrt{\int_0^{\infty} dr \int_0^{2\pi} d\varphi \, r \, \exp\left(-\frac{\beta}{2m} r^2 \right)} \\
&= \sqrt{\frac{2m}{\beta} \pi \int_0^{\infty} du \, \exp(-u)} \\
&= \sqrt{\frac{2m\pi}{\beta}}
\end{aligned}$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} dp_x\, e^{-\frac{\beta}{2m}p_x^2-\beta\omega y p_x} &= \int_{-\infty}^{\infty} dp_x\, e^{-\left(\frac{\beta}{2m}p_x^2+\beta\omega y p_x+\frac{m\beta}{2}\omega^2 y^2\right)+\frac{m\beta}{2}\omega^2 y^2} \\
&= e^{\frac{m\beta}{2}\omega^2 y^2}\cdot \int_{-\infty}^{\infty} dp_x\, e^{-\left(\sqrt{\frac{\beta}{2m}}p_x+\sqrt{\frac{m\beta}{2}}\omega y\right)^2} \\
&= \sqrt{\frac{2m}{\beta}} e^{\frac{m\beta}{2}\omega^2 y^2}\cdot \int_{-\infty}^{\infty} du\, e^{-u^2} \\
&= \sqrt{\frac{2m}{\beta}} e^{\frac{m\beta}{2}\omega^2 y^2}\cdot \sqrt{\pi} \\
&= \sqrt{\frac{2m\pi}{\beta}}\, e^{\frac{m\beta}{2}\omega^2 y^2}
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} dp_y\, e^{-\frac{\beta}{2m}p_y^2+\beta\omega x p_y} &= \int_{-\infty}^{\infty} dp_y\, e^{-\left(\frac{\beta}{2m}p_y^2-\beta\omega x p_y+\frac{m\beta}{2}\omega^2 x^2\right)+\frac{m\beta}{2}\omega^2 x^2} \\
&= e^{\frac{m\beta}{2}\omega^2 x^2}\cdot \int_{-\infty}^{\infty} dp_y\, e^{-\left(\sqrt{\frac{\beta}{2m}}p_y-\sqrt{\frac{m\beta}{2}}\omega x\right)^2} \\
&= \ldots \text{ equivalent to } p_x \ldots \\
&= \sqrt{\frac{2m\pi}{\beta}}\, e^{\frac{m\beta}{2}\omega^2 x^2}
\end{aligned}
$$

We get:

$$
Z_1 = \frac{1}{h^3}\left(\sqrt{\frac{2m\pi}{\beta}}\right)^3 \int d^3q\, e^{\frac{m\beta}{2}\omega^2 y^2}\, e^{\frac{m\beta}{2}\omega^2 x^2}
$$

Using cylindrical coordinates:

$$
\begin{aligned}
Z_1 &= \left(\frac{2m\pi}{h^2\beta}\right)^{3/2} \int_0^R dr \int_0^{2\pi} d\varphi \int_0^L dz\, r\, e^{\frac{m\beta}{2}\omega^2 r^2} \\
&= \left(\frac{2m\pi}{h^2\beta}\right)^{3/2} 2\pi L \int_0^{m\beta\omega^2 R^2/2} \frac{du}{2m\beta\omega^2 r/2}\, r\, e^u \\
&= \left(\frac{2m\pi}{h^2\beta}\right)^{3/2} \frac{2\pi L}{m\beta\omega^2}\cdot \left[e^u\right]_0^{m\beta\omega^2 R^2/2} \\
&= \left(\frac{2m\pi}{h^2\beta}\right)^{3/2} \frac{2\pi L}{m\beta\omega^2}\cdot \left(e^{m\beta\omega^2 R^2/2}-1\right)
\end{aligned}
$$

□

(b)

Using N identical particles we get:

$$
Z_N = \frac{1}{N!h^{3N}}\int d^{3N}q\, d^{3N}p\, e^{-\beta\sum_{i=1}^{N}\left(\frac{p_i^2}{2m}-\omega(x_i p_{y,i}-y_i p_{x,i})\right)}
$$

Momentum integration:

$$
\begin{aligned}
Z_N &= \frac{1}{N!h^{3N}} \int d^{3N}q \\
&\cdot \int_{-\infty}^{\infty} d^N p_x\, e^{-\beta \sum_{i=1}^N \left(\frac{p_{x,i}^2}{2m} + \omega y_i p_{x,i} \right)} \\
&\cdot \int_{-\infty}^{\infty} d^N p_y\, e^{-\beta \sum_{i=1}^N \left(\frac{p_{y,i}^2}{2m} - \omega x_i p_{y,i} \right)} \\
&\cdot \int_{-\infty}^{\infty} d^N p_z\, e^{-\beta \sum_{i=1}^N \frac{p_{z,i}^2}{2m}}
\end{aligned}
$$

with:

$$
\begin{aligned}
\int_{-\infty}^{\infty} dp_z\, e^{-\frac{\beta}{2m} \sum_{i=1}^N p_{z,i}^2} &= \left[\int_{-\infty}^{\infty} dp_z\, e^{-\frac{\beta}{2m} p_z^2} \right]^N \\
&= \left[\sqrt{\frac{2m\pi}{\beta}} \right]^N
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} d^N p_x\, e^{-\beta \sum_{i=1}^N \left(\frac{p_{x,i}^2}{2m} + \omega y_i p_{x,i} \right)} &= \left[\int_{-\infty}^{\infty} dp_x\, e^{-\frac{\beta}{2m} p_x^2 - \beta \omega y p_x} \right]^N \\
&= \left[\sqrt{\frac{2m\pi}{\beta}} \right]^N e^{N \frac{m\beta}{2} \omega^2 y^2}
\end{aligned}
$$

$$
\int_{-\infty}^{\infty} d^N p_y\, e^{-\beta \sum_{i=1}^N \left(\frac{p_{y,i}^2}{2m} - \omega x_i p_{y,i} \right)} = \left[\sqrt{\frac{2m\pi}{\beta}} \right]^N e^{N \frac{m\beta}{2} \omega^2 x^2}
$$

leads to:

$$
\begin{aligned}
Z_N &= \frac{1}{N!h^{3N}} \left[\sqrt{\frac{2m\pi}{\beta}} \right]^{3N} \int d^{3N}q\, e^{N \frac{m\beta}{2} \omega^2 y^2} e^{N \frac{m\beta}{2} \omega^2 x^2} \\
&= \frac{1}{N!h^{3N}} \left[\sqrt{\frac{2m\pi}{\beta}} \right]^{3N} \left[\int d^3 q\, e^{\frac{m\beta}{2} \omega^2 \left(x^2 + y^2 \right)} \right]^N
\end{aligned}
$$

Using result from **(a)**:

$$
Z_N = \frac{1}{N!} \left[\frac{2m\pi}{h^2 \beta} \right]^{\frac{3N}{2}} \left[\frac{2\pi L}{m\beta\omega^2} \right]^N \cdot \left(e^{m\beta\omega^2 R^2/2} - 1 \right)^N
$$

(c)

Definition of free energy:

$$F\left(T,V,N\right)=-\frac{1}{\beta}\ln Z_N\left(T,V\right)$$

inserting Z_N from **(b)**:

$$\begin{aligned}F\left(T,V,N\right) &= -\frac{1}{\beta}\ln\left[\frac{1}{N!}\left[\frac{2m\pi}{h^2\beta}\right]^{\frac{3N}{2}}\left[\frac{2\pi L}{m\beta\omega^2}\right]^N\cdot\left(e^{m\beta\omega^2R^2/2}-1\right)^N\right] \\ &= -\frac{1}{\beta}\left[-\ln N!+\frac{3}{2}N\cdot\ln\left[\frac{2m\pi}{h^2\beta}\right]+N\ln\left[\frac{2\pi L}{m\beta\omega^2}\right]+N\ln\left(e^{m\beta\omega^2R^2/2}-1\right)\right]\end{aligned}$$

(d)

Definition:

$$p=-\frac{\partial F}{\partial V}$$

Volume of a cylinder:

$$V=\pi R^2L$$

Since there is no direct V dependance in F and pressure is going to vary in dependance of R and L we calculate two different pressures p_R and p_L. Therefore one of them stays in as a variable.

$$p_R=-\frac{\partial F}{\partial V}\Big|_L=-\frac{\partial F}{\partial R}\frac{\partial R}{\partial V}\Big|_L=-\frac{1}{2\sqrt{\pi VL}}\frac{\partial F}{\partial R}\Big|_L=-\frac{1}{2\pi RL}\frac{\partial F}{\partial R}\Big|_L$$

inserting F from **(c)** $L=const.$:

$$\begin{aligned}p_R &= -\frac{1}{2\pi RL}\frac{\partial}{\partial R}\left(-\frac{1}{\beta}\left[-\ln N!+\frac{3}{2}N\cdot\ln\left[\frac{2m\pi}{h^2\beta}\right]+N\ln\left[\frac{2\pi L}{m\beta\omega^2}\right]+N\ln\left(e^{m\beta\omega^2R^2/2}-1\right)\right]\right) \\ &= \frac{1}{2\beta\pi RL}\left\{N\frac{\partial}{\partial R}\ln\left[\left(e^{m\beta\omega^2R^2/2}-1\right)\right]\right\} \\ &= \frac{N}{2\beta\pi RL}\left(\frac{e^{m\beta\omega^2R^2/2}}{e^{m\beta\omega^2R^2/2}-1}\cdot\frac{m\beta\omega^2}{2}\cdot 2R\right) \\ &= \frac{N}{2\pi L}\left(\frac{m\omega^2}{1-e^{-m\beta\omega^2R^2/2}}\right)\end{aligned}$$

This converges to $p_RV=Nk_BT$ for $\omega\to 0$, this can be seen using L'Hospital.
p_L:

$$p_R=-\frac{\partial F}{\partial V}\Big|_R=-\frac{\partial F}{\partial L}\frac{\partial L}{\partial V}\Big|_R=-\frac{1}{\pi R^2}\frac{\partial F}{\partial L}\Big|_R$$

with F, $R=const.$:

$$
\begin{aligned}
p_L &= -\frac{1}{\pi R^2}\frac{\partial}{\partial L}\left\{-\frac{1}{\beta}-\frac{N}{\beta}\ln\left[\frac{L\sqrt{m}}{h^3\omega^2 N}\cdot\left(e^{m\beta\omega^2R^2/2}-1\right)\right]-\frac{5N}{2\beta}\ln\left[\frac{2\pi}{\beta}\right]\right\} \\
&= \frac{N}{\beta\pi R^2}\frac{\partial}{\partial L}\left(\ln\frac{L\sqrt{m}}{h^3\omega^2 N}\right) \\
&= \frac{N}{\beta\pi R^2 L} \\
&= \frac{1}{\beta}\frac{N}{V}
\end{aligned}
$$

This is the ideal gas relation $p_L V = Nk_BT$, which seems likely for the z-axis, since there is no reason for any other dependance there (especially there is no dependance of ω).

Total force at outer walls:

$$
\begin{aligned}
F &= p\cdot A \\
&\text{with} \\
F_R &= p_R A_R \\
&= \frac{N}{2\pi L}\left(\frac{m\omega^2}{1-e^{-m\beta\omega^2R^2/2}}\right)\pi R^2 \\
F_R &= \frac{N}{2L}\left(\frac{m\omega^2R^2}{1-e^{-m\beta\omega^2R^2/2}}\right) \\
&\text{and} \\
F_L &= p_L A_L \\
&= \frac{1}{\beta}\frac{N}{\pi R^2 L}2\pi RL \\
F_L &= \frac{2N}{R\beta}
\end{aligned}
$$

4.4 (Doppler broadening)

Atoms with mass m, temperature T emit light in x-direction, moving with v_x leads to doppler effect:

$$\nu = \nu_0\left(1+\frac{v_x}{c}\right)$$

Using $v_x = \frac{p_x}{m}$ and Hamiltonian for free particle $H = \sum_{i=1}^{N}\frac{p_i^2}{2m}$:

$$\nu = \nu_0\left(1+\frac{p_x}{mc}\right)$$

(a)

Definition:

$$\langle A\rangle = \frac{\int d^{3N}q\int d^{3N}p\,A\left(\vec{q}_N,\vec{p}_N\right)e^{-\beta H(\vec{q}_N,\vec{p}_N)}}{\int d^{3N}q\int d^{3N}p\,e^{-\beta H(\vec{q}_N,\vec{p}_N)}}$$

inserting given problem conditions:

$$\begin{aligned}
\langle\nu\rangle &= \nu_0 \frac{\int d^{3N}q \int d^{3N}p \, \left(1+\frac{p_x}{mc}\right) e^{-\beta\sum_{i=1}^N \frac{p_i^2}{2m}}}{\int d^{3N}q \int d^{3N}p \, e^{-\beta\sum_{i=1}^N \frac{p_i^2}{2m}}} \\
&= \nu_0 \frac{\int d^{3N}q \int d^{3N}p \, e^{-\beta\sum_{i=1}^N \frac{p_i^2}{2m}}}{\int d^{3N}q \int d^{3N}p \, e^{-\beta\sum_{i=1}^N \frac{p_i^2}{2m}}} + \frac{\nu_0}{mc} \frac{\int d^{3N}q \int d^{3N}p \, p_x e^{-\beta\sum_{i=1}^N \frac{p_i^2}{2m}}}{\int d^{3N}q \int d^{3N}p \, e^{-\beta\sum_{i=1}^N \frac{p_i^2}{2m}}} \\
&= \nu_0
\end{aligned}$$

second term vanishes because of symmetry reasons (integrating over symmetric intervall, but p_x is odd, while exponent ($p^2 = p_x^2 + p_y^2 + p_z^2$) p_x^2 is even).

(b)

Definition:

$$\Delta\nu = \sqrt{\left\langle (\nu - \langle\nu\rangle)^2 \right\rangle} = \sqrt{\left\langle \nu^2 - 2\nu\langle\nu\rangle + \langle\nu\rangle^2 \right\rangle} = \sqrt{\langle\nu^2\rangle - \langle\nu\rangle^2}$$

$\left\langle \nu^2 \right\rangle$ calculation:

$$\begin{aligned}
\left\langle \nu^2 \right\rangle &= \nu_0^2 \frac{\int d^{3N}q \int d^{3N}p \, \left(1+\frac{p_x}{mc}\right)^2 e^{-\beta\sum_{i=1}^N \frac{p_i^2}{2m}}}{\int d^{3N}q \int d^{3N}p \, e^{-\beta\sum_{i=1}^N \frac{p_i^2}{2m}}} \\
&= \nu_0^2 + \frac{\nu_0^2}{(mc)^2} \frac{\int d^{3N}q \int d^{3N}p \, p_x^2 \, e^{-\beta\sum_{i=1}^N \frac{p_i^2}{2m}}}{\int d^{3N}q \int d^{3N}p \, e^{-\beta\sum_{i=1}^N \frac{p_i^2}{2m}}}
\end{aligned}$$

$\left(1+\frac{p_x}{mc}\right)^2 = \left(1 + 2\frac{p_x}{mc} + \left(\frac{p_x}{mc}\right)^2\right)$, first term leads to ν_0^2, while numerator and denominator integrals are the same, second term cancels because of symmetry. Integrals of last term have to be calculated:

denominator:

$$\int d^{3N}q \int d^{3N}p \, e^{-\beta\sum_{i=1}^N \frac{p_i^2}{2m}} = \int d^{3N}q \left[\int_{-\infty}^{\infty} dp \, e^{-\beta\frac{p^2}{2m}}\right]^{3N} = \left[\sqrt{\frac{2m\pi}{\beta}}\right]^{3N} \int d^{3N}q$$

numerator:

$$\begin{aligned}
\int d^{3N}q \int d^{3N}p \, p_x^2 \, e^{-\beta\sum_{i=1}^N \frac{p_i^2}{2m}} &= \int d^{3N}q \int_{-\infty}^{\infty} dp_x \, p_x^2 \, e^{-\frac{\beta}{2m}\sum_{i=1}^N p_{x,i}^2} \left[\int_{-\infty}^{\infty} dp \, e^{-\frac{\beta}{2m}p^2}\right]^{2N} \\
&= \left[\sqrt{\frac{2m\pi}{\beta}}\right]^{2N} \int d^{3N}q \left[\int_{-\infty}^{\infty} dp_x \, p_x^2 \, e^{-\frac{\beta}{2m}p_x^2}\right]^{N}
\end{aligned}$$

solving one dimensional integral:

$$
\begin{aligned}
\int_{-\infty}^{\infty} dp_x\, p_x^2\, e^{-\frac{\beta}{2m}p_x^2} &= \int_{-\infty}^{\infty} dp_x\, p_x \cdot p_x e^{-\frac{\beta}{2m}p_x^2} \\
&= \left[-p_x \cdot \frac{m}{\beta} e^{-\frac{\beta}{2m}p_x^2}\right]_{-\infty}^{\infty} + \frac{m}{\beta}\int_{-\infty}^{\infty} dp_x\, e^{-\frac{\beta}{2m}p_x^2} \\
&= \frac{m}{\beta}\sqrt{\frac{2m\pi}{\beta}}
\end{aligned}
$$

inserting in numerator integral:

$$
\begin{aligned}
\int d^{3N}q \int d^{3N}p\, p_x^2\, e^{-\beta\sum_{i=1}^{N}\frac{p_i^2}{2m}} &= \left[\sqrt{\frac{2m\pi}{\beta}}\right]^{2N}\left[\frac{m}{\beta}\sqrt{\frac{2m\pi}{\beta}}\right]^{N}\int d^{3N}q \\
&= \left[\sqrt{\frac{2m\pi}{\beta}}\right]^{3N}\left(\frac{m}{\beta}\right)^{N}\int d^{3N}q
\end{aligned}
$$

inserting numerator and denominator integral:

$$
\begin{aligned}
\left\langle \nu^2 \right\rangle &= \nu_0^2 + \frac{\nu_0^2}{(mc)^2}\frac{\left[\sqrt{\frac{2m\pi}{\beta}}\right]^{3N}\left(\frac{m}{\beta}\right)^{N}\int d^{3N}q}{\left[\sqrt{\frac{2m\pi}{\beta}}\right]^{3N}\int d^{3N}q} \\
&= \nu_0^2 \cdot \left(1 + \frac{m^N}{(mc)^2\beta^N}\right)
\end{aligned}
$$

This leads to:

$$
\Delta\nu = \sqrt{\langle\nu^2\rangle - \langle\nu\rangle^2} = \sqrt{\nu_0^2 + \frac{m^N}{(mc)^2\beta^N}\cdot\nu_0^2 - \nu_0^2} = \frac{m^{\frac{N}{2}}}{mc\beta^{\frac{N}{2}}}\cdot\nu_0
$$

with $N = 1$, this is what we need, since we only have a look at one "average"-particle:

$$
\Delta\nu = \sqrt{\frac{k_B T}{mc^2}}\cdot\nu_0
$$

(c)

Using Maxwell velocity distribution:

$$
w(\vec{v})\, d^3v = \left(\frac{m\beta}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{\beta m v^2}{2}\right) d^3v
$$

we get the velocity distribution for v_x:

$$
w(v_x)\, dv_x = \left(\frac{m\beta}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{\beta m v_x^2}{2}\right)\int_{-\infty}^{\infty} dv_y \exp\left(-\frac{\beta m v_y^2}{2}\right)\int_{-\infty}^{\infty} dv_z \exp\left(-\frac{\beta m v_z^2}{2}\right)
$$

$$
\begin{aligned}
&= \left(\frac{m\beta}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{\beta m v_x^2}{2}\right)\left[\int_{-\infty}^{\infty} dv \exp\left(-\frac{\beta m}{2} v^2\right)\right]^2 \\
&= \left(\frac{m\beta}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{\beta m v_x^2}{2}\right)\left[\sqrt{\frac{2\pi}{\beta m}}\right]^2 \\
&= \left(\frac{m\beta}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{\beta m v_x^2}{2}\right)
\end{aligned}
$$

while we got the direct correlation $I(\nu)\, d\nu = w(v_x)\, dv_x$ with

$$v_x = \frac{c}{\nu_0}(\nu - \nu_0) \quad \text{and} \quad dv_x = \frac{c}{\nu_0} d\nu$$

we get:

$$I(\nu)\, d\nu = \frac{1}{\sqrt{\pi}} \left(\frac{mc^2\beta}{2\nu_0^2}\right)^{\frac{1}{2}} \exp\left(-\frac{\beta mc^2 (\nu - \nu_0)^2}{2\nu_0^2}\right) d\nu$$

For the non relativistic case $k_B T \ll mc^2$, with $\beta = \frac{1}{k_B T}$, we have to consider, that $\beta mc^2 \to \infty$. While the squareroot term will diverge the exponential term will converge to zero. Substituting $x = \sqrt{\frac{mc^2\beta}{2\nu_0^2}}$ we get:

$$I(\nu) = \frac{x}{\sqrt{\pi}} \exp\left(-x^2 (\nu - \nu_0)^2\right)$$

which is a representation of the delta function for $x \to \infty$:

$$\lim_{x\to\infty} I(\nu) = \delta(\nu - \nu_0)$$

This means, we only get an intensity at $\nu_0 = \nu$, while ν_0 is the emitted frequency of the atoms. This means, the velocity of the atoms doesn't have any effect and the doppler effect doesn't occur. Meaning $v_x \ll c$:

$$\nu = \nu_0 \left(1 + \underbrace{\frac{v_x}{c}}_{\to 0}\right) \approx \nu_0$$