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3 Task Theoretical Physics VI - Statistics

3.1 (Discrete random variable)

We examine the x component of the angular momentum of some quantum system, that can only take on three values: $-\hbar, 0$ or $\hbar$. We know, that for a given state of the system:

$$\langle L_x \rangle = \frac{\hbar}{3} \quad \text{and} \quad \left\langle L_x^2 \right\rangle = \frac{2\hbar^2}{3}$$

(a)

We are meant to estimate the probability density $p(L_x)$ for the x component of the angular momentum. We can therefore use the fact, that:

$$\langle L_x \rangle = \sum_{i=0}^{2} L_{x,i} p(L_{x,i}) = \frac{\hbar}{3}$$

and

$$\left\langle L_x^2 \right\rangle = \sum_{i=0}^{2} L_{x,i}^2 p(L_{x,i}) = \frac{2\hbar^2}{3}$$

with $L_{x,0} = -\hbar$, $L_{x,1} = 0$ and $L_{x,2} = \hbar$.
Additionally we know the normalisation condition:

$$\sum_{i=0}^{2} p(L_{x,i}) = 1$$

Using this three conditions we receive the equationsystem:

$$\begin{aligned} -\hbar p(-\hbar) + 0p(0) + \hbar p(\hbar) &= \frac{\hbar}{3} \\ (-\hbar)^2 p(-\hbar) + 0^2 p(0) + \hbar^2 p(\hbar) &= \frac{2\hbar}{3} \\ p(-\hbar) + p(0) + p(\hbar) &= 1 \end{aligned}$$

This leads to the matrix form:

$$\begin{pmatrix} -\hbar & 0 & \hbar \\ \hbar^2 & 0 & \hbar^2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} p(-\hbar) \\ p(0) \\ p(\hbar) \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{3} \\ \frac{2\hbar^2}{3} \\ 1 \end{pmatrix}$$

We can calculate the first two rows, multiplying with $\hbar$ and then adding or subtracting them from another to get to:

$$\begin{pmatrix} -\hbar & 0 & \hbar & |\frac{\hbar}{3} \\ \hbar^2 & 0 & \hbar^2 & |\frac{2\hbar^2}{3} \\ 1 & 1 & 1 & |1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 2\hbar^2 & |\hbar^2 \\ 2\hbar^2 & 0 & 0 & |\frac{\hbar^2}{3} \\ 1 & 1 & 1 & |1 \end{pmatrix}$$

Therefore we already know:

$$
\begin{aligned}
p(\hbar) &= \frac{1}{2} \\
p(-\hbar) &= \frac{1}{6}
\end{aligned}
$$

Inserting this into the normalisation condition, we receive $p(0)$:

$$p(0) = 1 - p(\hbar) - p(-\hbar) = \frac{1}{3}$$

Therefore we get the probability density of:

$$p(L_x) = \frac{1}{6}\delta(L_x + \hbar) + \frac{1}{3}\delta(L_x) + \frac{1}{2}\delta(L_x - \hbar)$$

The sketch is presented in fig. 1.

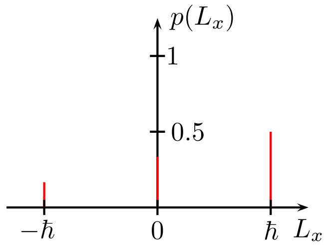

Figure 1: sketch of the probability density $p(L_x)$

(b)

For the cumulative function the following formula holds:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} p(y)\, dy$$

In our case this leads to:

$$F(L_x) = \int_{-\infty}^{L_x} p(y)\, dy = \int_{-\infty}^{L_x} \frac{1}{6}\delta(y + \hbar) + \frac{1}{3}\delta(y) + \frac{1}{2}\delta(y - \hbar)\, dy$$

This leads to:

$$F(L_x) = \begin{cases} 0 & \text{for } L_x < -\hbar \\ \frac{1}{6} & \text{for } -\hbar \leq L_x < 0 \\ \frac{1}{2} & \text{for } 0 \leq L_x < \hbar \\ 1 & \text{for } L_x \geq \hbar \end{cases}$$

The sketch can be found in fig. 2. While there are indeed only three values, that can be adopted, the sketch shouldn't be joined, because the values are discrete. But anyway, if there would be values in between, they would have this value, so it is sketched in joined style to express this fact.

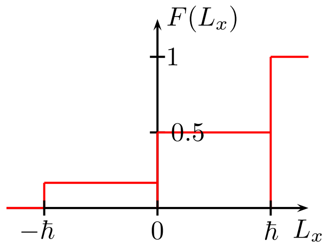

Figure 2: sketch of the cumulative function $F(L_x)$

3.2 (Classical particles in a box)

(a)

We are considering a classical particle of mass m moving in one dimension. The coordinate is denoted by x, while the momentum is denoted by p. The particle is confined to a box, being located between $x = 0$ and $x = L$. We know, that the energy of the particle lies between E and $E + \delta E$ and we are meant to draw the classical phase space for the particle.

We first have to consider the hamiltonian for the problem.

$$H(x, p) = \frac{p^2}{2m} + V(x)$$

While there is only the boundary condition of $x = 0$ and $x = L$ we find the potential

$$V(x) = \begin{cases} 0 & 0 \leq x \leq L \\ \infty & \text{other cases} \end{cases} \Leftrightarrow V(x) = \frac{1 - (\Theta(x) - \Theta(x - L))}{(\Theta(x) - \Theta(x - L))}$$

Now we have to look at the energy of the particle, which is given with:

$$H(x, p) = E \quad \text{and} \quad H(x, p) = E + \delta E$$

the potential just gives us a condition where the particle might be, so it is between $x = 0$ and $x = L$, while other x-space isn't accessible, even if we increased E this wouldn't be possible. This means that the x value doesn't influence the energy value, while the potential is 0, as long as $0 \leq x \leq L$, but in the other x-cases the equation cannot be fullfilled, because the potential is infinite. The $T = \frac{p^2}{2m}$ component therefore will create the shape of our phase space graph. While $\frac{p^2}{2m} = E \Leftrightarrow \mp p = \sqrt{\frac{E}{2m}}$ is only created by one positive p value (and we are just looking at positive ones for the phase space) and $\frac{p^2}{2m} = E + \delta E \Leftrightarrow \mp p = \sqrt{\frac{E + \delta E}{2m}}$ too, while the start and ending point in x-space will be $x = 0$ and $x = L$, we just get to straight lines for the energy condition borders. The regions between E and $E + \delta E$ are accessible to the particle, while other regions are not.

The sketch can be found in fig. 3. (We only sketch the positive part of the graph, since all information needed is contained in there, for sure the impulse can be negative too, since we got the p^2 term, therfore we get with x as a mirror plane the same bar at the $-p$-side).

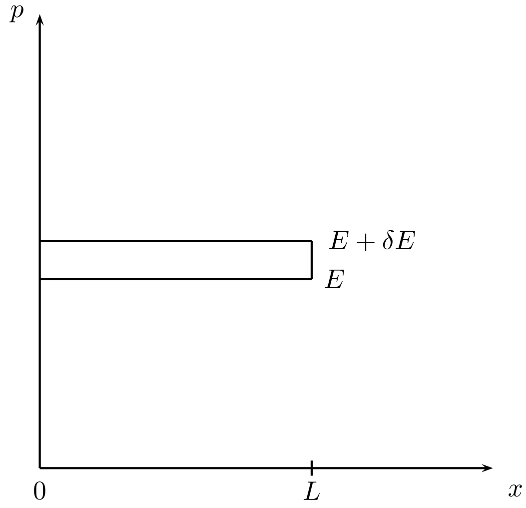

Figure 3: sketch of phase space for one particle

(b)

This time we are considering two weakly interacting classical particle of mass m with positions x_1, x_2 and momenta p_1, p_2 in the same box. The particles are confined to the box, being located between $x = 0$ and $x = L$. We know, that the total energy of the system lies between E and $E + \delta E$ and we are meant to draw the classical phase space for x_1, x_2 and p_1, p_2.

We again have to consider the hamiltonian for the problem.

$$H\left(x_1, x_2, p_1, p_2\right) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V\left(x_1, x_2\right)$$

While there is only the boundary condition of $x = 0$ and $x = L$ we find the potential

$$V\left(x\right) = \begin{cases} 0 & 0 \leq x \leq L \\ \infty & \text{other cases} \end{cases} \Leftrightarrow V\left(x\right) = \frac{1 - \left(\Theta\left(x\right) - \Theta\left(x - L\right)\right)}{\left(\Theta\left(x\right) - \Theta\left(x - L\right)\right)}$$

which is equal for $x = x_1$ and $x = x_2$.

Now we have to look at the energy of the system, which is given by:

$$H\left(x_1, x_2, p_1, p_2\right) = E \quad \text{and} \quad H\left(x_1, x_2, p_1, p_2\right) = E + \delta E$$

While the coordinates x_1 and x_2 doesn't have a real dependance, since they are only part of the potential, which only leads to a straight line, we can immediately draw the x_1, x_2-plane. The energy-condition is meant to be fullfilled by the p_1, p_2-components, if we use the 4 dimensional space. While the first particle with coordinate x_1 can be between $x_1 = 0$ and $x_1 = L$, which also is possible for the second particle using x_2-axis, what we get is a box.

The p_1, p_2-plane is a little bit harder, this time we got the condition

$$E = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} \quad \text{and} \quad E + \delta E = \frac{p_1^2}{2m} + \frac{p_2^2}{2m}$$

which will be fullfilled by various p_1 and p_2 combinations. It seems likely that we will receive some part of a circle for both the conditions, resulting into an arc of a ring for the accessible p_1, p_2 phase space. While we only consider positive values, we will only see an arc of the circle. The negative values of p would only make sense, if we would give the negative values the opposite direction of the positive values. This can be done, but it should be enough to represent the arc of the ring to see the phase space figure.

The sketch for the x_1, x_2-plane be found in fig. 4, while the sketch for the p_1, p_2-plane can be found in fig. 5. (We only sketch the positive part of the graph, since all information needed is contained in there, for sure the impulse can be negative too, since we got the p^2 term, meaning we can see half a ring).

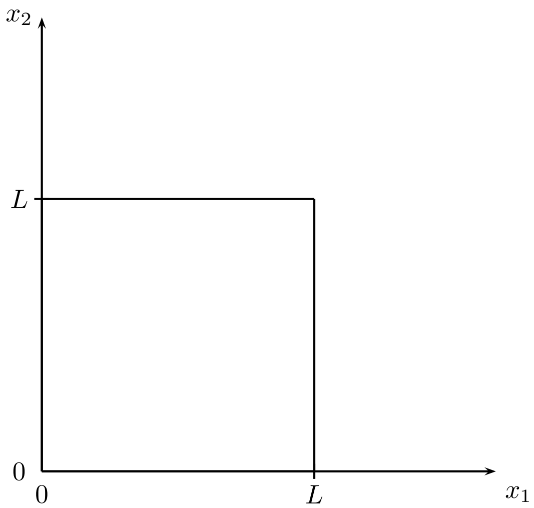

Figure 4: sketch of phase space for two particles, x_1, x_2-plane

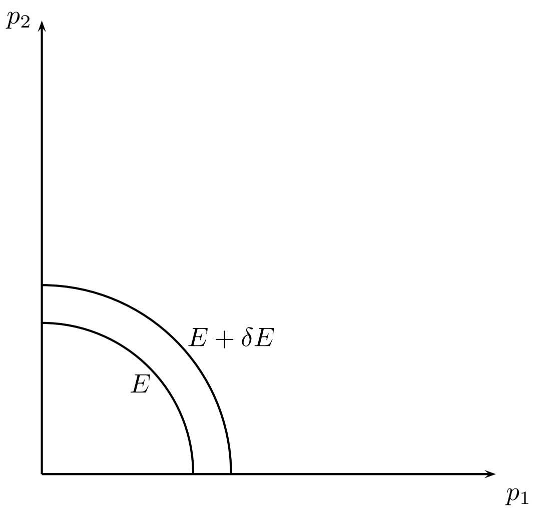

Figure 5: sketch of phase space for two particles, p_1, p_2-plane

3.3 (Non-interacting fermions)

The joint probability for finding particle 1 at $x = x_1$ and particle 2 at $x = x_2$ for two non-interacting spinless fermions in a one-dimensional harmonic trap is:

$$p(x_1, x_2) = \frac{1}{\pi x_0^2} \left(\frac{x_2 - x_1}{x_0} \right)^2 \exp\left[-\frac{x_1^2 + x_2^2}{x_0^2} \right]$$

with $x_0 \neq 0$ the characteristic distance.

(a)

The sketch of the x_1, x_2-plane with the joint probability can be found in fig. 6.

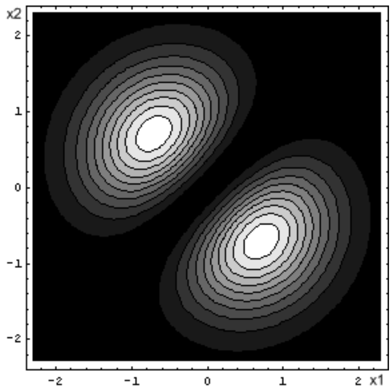

Figure 6: sketch of the joint probability, white means maximal, while black means minimal

(b)

To find the probability for finding particle at position 1 and position $x = x_1$, we have to integrate the second particle over the whole space, since we don't want the condition of particle 2 to be at position x_2 any more, particle 1 should not be influenced by particle 2. We therefore get:

$$p(x_1) = \int_{-\infty}^{\infty} p(x_1, x_2)\, dx_2 = \frac{1}{\pi x_0^4} \int_{-\infty}^{\infty} (x_2 - x_1)^2 \exp\left[-\frac{x_1^2 + x_2^2}{x_0^2} \right] dx_2$$

We can simplify this integral into 3 easier integrals:

$$\begin{aligned} p(x_1) &= \frac{1}{\pi x_0^4} \exp\left[-\frac{x_1^2}{x_0^2} \right] \cdot \int_{-\infty}^{\infty} x_2^2 \exp\left[-\frac{x_2^2}{x_0^2} \right] dx_2 \text{ (first)} \\ &- \frac{2x_1}{\pi x_0^4} \exp\left[-\frac{x_1^2}{x_0^2} \right] \cdot \int_{-\infty}^{\infty} x_2 \exp\left[-\frac{x_2^2}{x_0^2} \right] dx_2 \text{ (middle)} \\ &+ \frac{x_1^2}{\pi x_0^4} \exp\left[-\frac{x_1^2}{x_0^2} \right] \cdot \int_{-\infty}^{\infty} \exp\left[-\frac{x_2^2}{x_0^2} \right] dx_2 \text{ (second)} \end{aligned}$$

With symmetry considerations the middle integral cancels because we integrate over a symmetric interval and the integrand consists of an even and an odd term. So we only have to solve the other two integrals, from which the second one is the easier one, while the first one can be brought into the second integrals form through partial integration. We first solve the second one:

$$\begin{aligned}
\int_{-\infty}^{\infty} \exp\left[-\frac{1}{x_0^2}x_2^2\right] dx_2 &= \sqrt{\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \exp\left[-\frac{1}{x_0^2}\left(x^2+y^2\right)\right]} \\
&= \sqrt{\int_0^{\infty} dr \int_0^{2\pi} d\varphi \, r \, \exp\left(-\frac{1}{x_0^2}r^2\right)} \\
&= \sqrt{x_0^2\pi \int_0^{\infty} du \, \exp\left(-u\right)} \\
&= \sqrt{x_0^2\pi} \\
&= x_0\sqrt{\pi}
\end{aligned}$$

Now we use the partial integration method to reduce the first one to the form of the second one:

$$\begin{aligned}
\int_{-\infty}^{\infty} x_2^2 \exp\left[-\frac{x_2^2}{x_0^2}\right] dx_2 &= \left[x\cdot\left(-\frac{x_0^2}{2}\exp\left[-\frac{x_2^2}{x_0^2}\right]\right)\right]_{-\infty}^{\infty} + \frac{x_0^2}{2}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{x_0^2}x_2^2\right] dx_2 \\
&= \frac{1}{2}x_0^2 \cdot \int_{-\infty}^{\infty} \exp\left[-\frac{1}{x_0^2}x_2^2\right] dx_2 \\
&= \frac{1}{2}x_0^2 \cdot x_0\sqrt{\pi}
\end{aligned}$$

Inserting the results for the integrals into the equation for $p\left(x_1\right)$ we receive:

$$\begin{aligned}
p\left(x_1\right) &= \frac{1}{\pi x_0^4}\exp\left[-\frac{x_1^2}{x_0^2}\right]\cdot\frac{1}{2}x_0^3\sqrt{\pi} + \frac{x_1^2}{\pi x_0^4}\exp\left[-\frac{x_1^2}{x_0^2}\right]\cdot x_0\sqrt{\pi} \\
&= \frac{1}{\pi x_0^4}\exp\left[-\frac{x_1^2}{x_0^2}\right]\cdot\left(\frac{1}{2}x_0^3\sqrt{\pi} + x_1^2 x_0\sqrt{\pi}\right)
\end{aligned}$$

finally we get:

$$p\left(x_1\right) = \int_{-\infty}^{\infty} p\left(x_1, x_2\right) \, dx_2 = \frac{1}{2\sqrt{\pi}x_0^3}\left(x_0^2 + 2x_1^2\right)\exp\left[-\frac{x_1^2}{x_0^2}\right]$$

While we can simply substitute $x_2 \rightarrow x_1$ and get the same starting equation, we immediately see, that the probability for finding particle 2 at the position $x = x_2$ is:

$$p\left(x_2\right) = \int_{-\infty}^{\infty} p\left(x_1, x_2\right) \, dx_1 = \frac{1}{2\sqrt{\pi}x_0^3}\left(x_0^2 + 2x_2^2\right)\exp\left[-\frac{x_2^2}{x_0^2}\right]$$

From this point we see, that the particles are statistically dependent, since the correlation criterium for independance:

$$p(x_1, x_2) = p(x_1) \cdot p(x_2)$$

doesn't hold for this case, since

$$\begin{aligned} p(x_1) \cdot p(x_2) &= \frac{1}{\pi x_0^2}\left(\frac{1}{2}+\frac{x_1^2}{x_0^2}\right)\left(\frac{1}{2}+\frac{x_2^2}{x_0^2}\right)\exp\left[-\frac{x_1^2+x_2^2}{x_0^2}\right] \\ &\neq \frac{1}{\pi x_0^2}\left(\frac{x_2-x_1}{x_0}\right)^2 \exp\left[-\frac{x_1^2+x_2^2}{x_0^2}\right] \\ &= p(x_1, x_2) \\ \Rightarrow p(x_1, x_2) &\neq p(x_1) \cdot p(x_2) \end{aligned}$$

Therefore the particles are statistically dependent.

(c)

The conditional probability density $p(x_1|x_2)$ is defined as:

$$p(x_1|x_2) = \frac{p(x_1, x_2)}{p(x_2)}$$

We can use $p(x_2)$ from **(b)** and therefore we get:

$$p(x_1|x_2) = \frac{\frac{1}{\pi x_0^2}\left(\frac{x_2-x_1}{x_0}\right)^2 \exp\left[-\frac{x_1^2+x_2^2}{x_0^2}\right]}{\frac{1}{2\sqrt{\pi}x_0^3}\left(x_0^2+2x_2^2\right)\exp\left[-\frac{x_2^2}{x_0^2}\right]} = \frac{2\sqrt{\pi}x_0^3\left(\frac{x_2-x_1}{x_0}\right)^2 \exp\left[-\frac{x_1^2+x_2^2}{x_0^2}\right]\exp\left[\frac{x_2^2}{x_0^2}\right]}{\pi x_0^2\left(x_0^2+2x_2^2\right)}$$

This can be further simplified to:

$$p(x_1|x_2) = \frac{2\left(x_2-x_1\right)^2}{\sqrt{\pi}x_0\left(x_0^2+2x_2^2\right)}\exp\left[-\frac{x_1^2}{x_0^2}\right]$$

The sketch of the x_1, x_2-plane with the conditional probability can be found in fig. 7.

3.4 (Waiting times)

Each hour 12 buses and 12 cars passing by on average. The buses are scheduled and appear exactly every 5 minutes, while the cars appear at random. The probability that a car passes in a time interval dt is given by $\frac{dt}{\tau}$ with $\tau = 5\,\text{minutes}$. While we got lots of "experiments", but only few events we can use the poisson distribution:

$$p(n) = \frac{\lambda^n}{n!}e^{-\lambda} = \frac{\left(\frac{dt}{\tau}\right)^n}{n!}e^{-\frac{dt}{\tau}}$$

with $\langle n \rangle = \lambda = \frac{dt}{\tau}$ the average appearance frequency of occurence (Auftretenshäufigkeit) of an event.

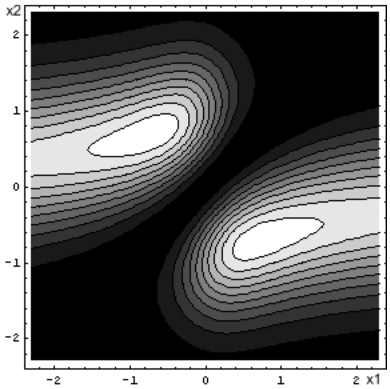

Figure 7: sketch of $p\left(x_1|x_2\right)$, white means maximal, while black means minimal

(a)

We are meant to verify, that there is an average number of 12 cars passing in an hour. We first calculate the expectation value:

$$\langle n\rangle = \sum_{n=0}^{\infty} np\left(n\right) = \sum_{n=0}^{\infty} n\frac{\lambda^n}{n!}e^{-\lambda} = \lambda\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!}e^{-\lambda} = \lambda\underbrace{\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda}}_{=1} = \lambda$$

The average appearance frequency of occurence of an event is given by

$$\lambda = \frac{dt}{\tau}$$

we simply have to insert $\tau = 5\,\text{minutes}$ and $dt = 1\,\text{h} = 60\,\text{minutes}$ and we get:

$$\lambda = \frac{60\,\text{minutes}}{5\,\text{minutes}} = 12$$

$\square$

(b)

We have a look at a randomly chosen 10 minute time interval, this means $dt = 10\,\text{minutes}$. The probability of $P_{\text{bus}}\left(n\right) = p_b\left(n\right)$ and $P_{\text{car}}\left(n\right) = p_c\left(n\right)$ are meant to be calculated, while n is the number of vehicles of the demanded type that pass. For the probability density of the buses we can write:

$$p_b\left(n\right) = \delta_{2n}$$

where $\delta_{2n} = 1$ for $n = 2$ and 0 for any other value. This results from the following thoughts.

The probability for at least 2 buses to pass is:

$$p_{b,\text{atleast}}\left(2\right) = 1$$

meaning, that you will see 2 buses in every 10 minute interval, also we know, that the chance to see $k > 3$ buses in the 10 minute interval is:

$$p_b(k) = 0$$

So we only need to have a look at the special case, where there are 2 or 3 buses, while there is only one moment in time, where you can see 3 buses. We will neglect that moment, because it should be infinitesimal in this model. We also don't have to consider the case of just seeing 1 bus, because the time interval is of that choice, that we will at least see 2 of them. Using the formula $p_b(n) = \delta_{2n}$ for the probabilty density of the buses with $dt = 10\,\text{minutes}$ for sure leads to the probability table:

n	$p_b(n)$
0	0
1	0
2	1
>2	0

Now we consider the case of the cars. While we use the poisson distribution we have:

$$p_c(n) = \frac{\left(\frac{dt}{\tau}\right)^n}{n!} e^{-\frac{dt}{\tau}} = \frac{2^n}{n!} e^{-2}$$

We get the following probability table:

n	$p_c(n)$	$\sum_n p_c(n)$
0	0.135	0.135
1	0.271	0.406
2	0.271	0.677
3	0.180	0.857
4	0.090	0.947
5	0.036	0.983
6	0.012	0.995
7	0.003	0.999
8	0.001	0.9998

We are now asked to calculate the mean and the variance of those distributions. While the mean of $p_b(n)$ can be given at instant with

$$\langle n \rangle_b = 2$$

the variance can be given with

$$\sigma_b^2 = 0$$

this follows because there is just one possibility, which is $n = 2$ and no other events will occur.

While the mean value and the variance are equal for a poisson distribution, which can be shown in some lines:

$$
\begin{aligned}
\sigma^2 &= \langle n - \langle n \rangle \rangle^2 \\
&= \langle n - \lambda \rangle^2 \\
&= \sum_{n=0}^{\infty} (n-\lambda)^2 \frac{\lambda^n}{n!} e^{-\lambda} \\
&= \sum_{n=0}^{\infty} \left(n^2 - 2n\lambda + \lambda^2\right) \frac{\lambda^n}{n!} e^{-\lambda} \\
&= \sum_{n=0}^{\infty} n^2 \frac{\lambda^n}{n!} e^{-\lambda} - 2\lambda^2 + \lambda^2 \\
&= \sum_{n=0}^{\infty} \left(n\left(n-1\right) + n\right) \frac{\lambda^n}{n!} e^{-\lambda} - \lambda^2 \\
&= \sum_{n=0}^{\infty} n\left(n-1\right) \frac{\lambda^n}{n!} e^{-\lambda} + \lambda - \lambda^2 \\
&= \lambda^2 \sum_{n=2}^{\infty} \frac{\lambda^{n-2}}{(n-2)!} e^{-\lambda} + \lambda - \lambda^2 \\
&= \lambda
\end{aligned}
$$

where we used the normalisation condition $1 = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda}$.
We now easily get the values for $p_c\left(n\right)$:

$$
\begin{aligned}
\langle n \rangle_c &= \frac{dt}{\tau} \\
&= \frac{10\,\text{minutes}}{5\,\text{minutes}} \\
&= 2
\end{aligned}
$$

and the variance is also found to be $\sigma_c^2 = 2$.

(c)

We this time seek for the probability distributions $P_{\text{bus}}\left(\Delta t\right) = p_b\left(\Delta t\right)$ and $P_{\text{car}}\left(\Delta t\right) = p_c\left(\Delta t\right)$ for the time interval Δt between two successive buses and cars. We presume that the first bus is coming, when we start our time. Therefore we will not be able to fullfill the 2 bus condition under 5 minutes, while it is fullfilled after 5 minutes. We get the probability distribution:

$$
p_b\left(\Delta t\right) = \delta\left(5\,\text{minutes} - \Delta t\right)
$$

The mean value is:

$$
\begin{aligned}
\langle \Delta t \rangle &= \int_0^{\infty} \Delta t\, p_b\left(\Delta t\right)\, d\Delta t \\
&= \int_0^{\infty} \Delta t\, \delta\left(5\,\text{minutes} - \Delta t\right)\, d\Delta t \\
&= 5\,\text{minutes}
\end{aligned}
$$

and the variance:

$$\langle\Delta t^2\rangle - \langle\Delta t\rangle^2 = \int_{-\infty}^{\infty} (\Delta t)^2 \, \delta(5\,\text{minutes} - \Delta t) \, d\Delta t - (5\,\text{minutes})^2 = (25-25)\,\text{minutes}^2 = 0\,\text{minutes}^2$$

We can now use the condition $n = 0$ for the poisson distribution of the car case, while for t-dependancy the distribution turns into a continuous one:

$$p_c(0,t) = Ne^{-\frac{t}{\tau}}$$

The reason for using $n = 0$ follows from the average time we have to wait between two events, which is exactly what we seek for, the interval between two events.

Unluckily the distribution isn't normalised for time, that's why we inserted the normalisation constant N, so we can calculate the normalisation with $1 = \int_0^\infty p(t)\,dt$:

$$\begin{aligned} \frac{1}{N} &= \int_0^\infty e^{-\frac{t}{\tau}} dt \\ &= \tau \\ \Rightarrow N &= \frac{1}{\tau} \end{aligned}$$

While we will need the integral:

$$\int_0^\infty t^n e^{-\alpha t}\,dt$$

several times, we will get a value for this in dependance of n, while we start out with partial integration:

$$\begin{aligned} \int_0^\infty t^n e^{-\alpha t}\,dt &= \underbrace{\left[-t^n \frac{1}{\alpha} e^{-\alpha t}\right]_0^\infty}_{0} + \frac{n}{\alpha}\int_0^\infty t^{n-1} e^{-\alpha t}\,dt \\ &= 0 + \frac{n(n-1)}{\alpha^2}\int_0^\infty t^{n-2} e^{-\alpha t} dt \\ &\quad \ldots \text{ again } n-2 \text{ times} \\ &= \frac{n!}{\alpha^n}\int_0^\infty dt\; e^{-\alpha t} \\ &= \frac{n!}{\alpha^{n+1}}. \end{aligned}$$

which leeds to

$$\int_0^\infty t^n e^{-\alpha t}\,dt = \frac{n!}{\alpha^{n+1}}.$$

Which we can show really fast with induction. The induction start for $n = 0:$

$$\begin{aligned} \int_0^\infty dt\, t^0 e^{-\alpha t} &= \int_0^\infty dt\, e^{-\alpha t} \\ &= \frac{1}{\alpha} = \frac{0!}{\alpha^{0+1}} \checkmark \end{aligned}$$

Induction step:

$$n \to n+1: \ \frac{n!}{\alpha^{n+1}} \to \frac{(n+1)!}{\alpha^{n+2}}$$

is given by

$$\begin{aligned}
\int_0^\infty dt\, t^{n+1} e^{-\alpha t} &= \left[-\frac{1}{\alpha} t^{n+1} e^{-\alpha t}\right]_0^\infty + \frac{n+1}{\alpha} \underbrace{\int_0^\infty dt\, t^n e^{-\alpha t}}_{=\frac{n!}{\alpha^{n+1}}} \\
&= \frac{(n+1)!}{\alpha^{n+2}} \\
&\Rightarrow \int_0^\infty dt\, t^n e^{-\alpha t} = \frac{n!}{\alpha^{n+1}}
\end{aligned}$$

□

The probability function with the normalisation constant then is (t in minutes):

$$p_c(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}$$

This leads to the mean value (using $\int_0^\infty dt\, t^n e^{-\frac{t}{\tau}} = n! \tau^{n+1}$):

$$\begin{aligned}
\langle t \rangle_c &= \int_0^\infty t\, p(t)\, dt \\
&= \frac{1}{\tau} \int_0^\infty t e^{-\frac{t}{\tau}} dt \\
&= \frac{1}{\tau} \cdot \tau^2 \\
&= \tau \\
&= 5\,\text{minutes}
\end{aligned}$$

We get the variance:

$$\left\langle t^2 \right\rangle - \langle t \rangle^2 \Big|_c = \frac{1}{\tau} \int_0^\infty t^2 e^{-\frac{t}{\tau}} dt - \tau^2 = \frac{1}{\tau} 2\tau^3 - \tau^2 = \tau^2 = 25\,\text{minutes}^2.$$

(d)

We now want to have a look at the probability distribution of the time, which another observer who arrives at the road at a randomly chosen time has to wait for the first bus to arrive. The same interest is inhabited in our minds for cars. While for the cars this is an easy task, since the person arrives at random and just has to wait, which is pretty much the same problem like **c)**, because he has to wait again, as long as there is no event. Therefore we get the result:

$$\begin{aligned}
p_c(t) &= \frac{1}{\tau} e^{-\frac{t}{\tau}} \\
\langle t \rangle_c &= \tau \\
\left\langle t^2 \right\rangle - \langle t \rangle^2 \Big|_c &= \tau^2
\end{aligned}$$

For the buses this time we know, that the maximum waiting time, till a bus will arrive is 5 minutes, while we don't know for sure, how long it really is going to be, since the person arrives at random, we at least know, that the probability to arrive at any of the minutes is $\frac{1}{5}$. The probability distribution for the bus therefore is:

$$p_b\left(t\right) = \frac{1}{5\,\text{minutes}} \text{ and } t \in [0,5]$$

We can calculate the mean value with:

$$\begin{aligned}
\langle t\rangle_b &= \int_{0\,\text{minutes}}^{5\,\text{minutes}} tp_b\left(t,\phi\right)dt \\
&= \int_{0\,\text{minutes}}^{5\,\text{minutes}} t\frac{1}{5\,\text{minutes}}dt \\
&= \frac{1}{5}\cdot\frac{5^2}{2}\,\text{minutes} \\
&= 2.5\,\text{minutes}
\end{aligned}$$

This seems most likely for the mean value. The variance can be calculated to be:

$$\begin{aligned}
\left\langle t^2\right\rangle - \langle t\rangle^2\Big|_b &= \frac{1}{5\,\text{minutes}}\int_{0\,\text{minutes}}^{5\,\text{minutes}} t^2dt - \frac{25}{4}\,\text{minutes}^2 \\
&= \frac{1}{5}\frac{5^3}{3}\,\text{minutes}^2 - \frac{25}{4}\,\text{minutes}^2 \\
&= \frac{100}{12}\,\text{minutes}^2 - \frac{75}{12}\,\text{minutes}^2 \\
&= \frac{25}{12}\,\text{minutes}^2
\end{aligned}$$

Therefore the bus values are:

$$\begin{aligned}
p_b\left(t,\phi\right) &= \delta\left(5\,\text{minutes} - \phi - t\right) \\
\langle t\rangle_b &= 2.5\,\text{minutes} \\
\left\langle t^2\right\rangle - \langle t\rangle^2\Big|_b &= \frac{25}{12}\,\text{minutes}^2
\end{aligned}$$

We can now compare the results from **c)** with **d)**. While for the at random appearing cars the waiting time for an randomly appearing observer and the waiting time between two events is equal, the waiting time for the bus between two buses is two times higher then for the observer waiting time, that is appearing at random. The variance now for the buses is relevant, while in **c)** it didn't have a value. This results from the randomness of the appearance of the obeserver, while before everything was well planned and known.