Theoretical Physics VI: Statistical Physics - Theory of Heat

Problem Set 13 (optional)

due: 6. 2. 2008, 10:15 am

This is an optional problem,
i.e. its points will be added to your total homework score up to 300 pts = 12×25 pts.

Problem 12.3 *Exact solution of the Ising model in $d = 1$* (15 pts.)
The Ising model in a one-dimensional lattice is specified by the Hamiltonian,

$$\hat{\mathbf{H}} = -J \sum_{i=1}^{N-1} \hat{\mathbf{S}}_i \hat{\mathbf{S}}_{i+1} - g\mu_B B_0 \sum_{i=1}^{N} \hat{\mathbf{S}}_i \,,$$

where $\vec{B} = B_0 \vec{e}_z$ is an external magnetic field, and $S_i = \pm\frac{1}{2}$. J, μ_B and B_0 are constant, with $J > 0$. Periodic boundary conditions, $\hat{\mathbf{S}}_{N+1} = \hat{\mathbf{S}}_1$, are assumed; this will not affect results in the thermodynamic limit, $N \to \infty$.

(a) Show that the partition function can be written as

$$\begin{aligned} Z_N(T, B_0) &= \sum_{S_1, S_2, \ldots, S_N} e^{\beta \sum_i (g\mu_B B_0 S_i + J S_i S_{i+1})} \\ &= \sum_{S_1, S_2, \ldots, S_N} \tau(S_1, S_2) \tau(S_2, S_3) \ldots \tau(S_N, S_1) \end{aligned}$$

and determine $\tau(S_i, S_{i+1})$.

(b) The sum is evaluated by introducing the transfer matrix,

$$T = \begin{pmatrix} \tau(\frac{1}{2}, \frac{1}{2}) & \tau(\frac{1}{2}, -\frac{1}{2}) \\ \tau(\frac{1}{2}, -\frac{1}{2}) & \tau(-\frac{1}{2}, -\frac{1}{2}) \end{pmatrix} .$$

Show that

$$\sum_{S_i, S_j, S_k} \tau(S_i, S_j) \tau(S_j, S_k) = \sum_{S_i, S_k} \left(T^2\right)_{S_i, S_k} \,,$$

with $(T^2)_{S_i, S_k} \equiv \sum_{S_j = \pm\frac{1}{2}} \tau(S_i, S_j) \tau(S_j, S_k)$.

(c) Moreover, show that the partition function can be expressed in terms of the two eigenvalues, $\lambda_{1,2}$, of the transfer matrix,

$$Z_N(T, B_0) = \lambda_1^N + \lambda_2^N \,.$$

Consider the limit $\lambda_2/\lambda_1 \ll 1$: Calculate

(d) the free energy, $F(T, B_0)$,

(e) the magnetization, $M(T, B_0)$ and the isothermal susceptibility, $\chi_T(B_0)$.