Theoretical Physics VI: Statistical Physics - Theory of Heat

Problem Set 12

due: 23. 1. 2008, 10:15 am

This is the last obligatory homework set.
There will be one optional homework set due 6 February, 2008,
which can be used to improve the homework grade.

Problem 12.1 *Bose-Einstein condensation in $d = 1, 2$?* (4 pts.)
Bose-Einstein condensation occurs when the chemical potential becomes equal to the single-particle ground state energy ϵ_0. For an ideal Bose gas without external potential $\epsilon_0 = 0$. In order to see whether $\mu = 0$ leads to physically sensible results, verify whether the number of particles,

$$N = \sum_p \frac{1}{e^{\beta(\epsilon_p - \mu)} - 1} = \left(\frac{L}{2\pi\hbar}\right)^d \int \frac{d^d p}{e^{\beta(p^2/(2m) - \mu)} - 1} ,$$

converges for spatial dimension $d = 1, 2$.
Hint: Convert the integral over momentum into an integral over energy.

Problem 12.2 *Bose-Einstein condensation in a harmonic potential* (12 pts.)
Bose-Einstein condensation of a dilute gas of bosonic atoms of mass m is achieved in the laboratory by confining them in a magnetic trap. The trap can be approximated by a harmonic potential which gives rise to the single-particle Hamiltonian,

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\hat{\mathbf{p}}_x^2 + \hat{\mathbf{p}}_y^2 + \hat{\mathbf{p}}_z^2\right) + \frac{1}{2} m \left(\omega_x^2 \hat{\mathbf{x}}^2 + \omega_y^2 \hat{\mathbf{y}}^2 + \omega_z^2 \hat{\mathbf{z}}^2\right) ,$$

where ω_x, ω_y, ω_z represent the trap frequencies in different directions. Let $\bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3}$.

(a) For large enough energies $\epsilon \gg \hbar\bar{\omega}$, the discrete energy eigenvalues may be approximated by a continuous distribution with a density of states $D(\epsilon)$, where $D(\epsilon)\Delta\epsilon$ is the number of energy eigenstates in the energy interval $(\epsilon, \epsilon + \Delta\epsilon)$. Find an expression for $D(\epsilon)$.

(b) Using this approximation, derive an equation for the chemical potential, given that there are a total of N atoms in the trap. Use this result to obtain the Bose-Einstein condensation temperature T_c. Give a physical interpretation of this result by comparing the thermal de Broglie wavelength to the characteristic size of the trap. What is the condition that needs to be satisfied for the approximation of the continuous density of states to yield an accurate value of T_c?

(c) Find an expression for the condensate density at a temperature $T < T_c$ using the approximation of the continuous density of states.

(d) In a typical experimental setup, ^{87}Rb atoms are confined in a trap with frequencies $\frac{1}{2\pi}(\omega_x, \omega_y, \omega_z) = (250, 670, 7)$ Hz and $N = 10^7$ atoms. Find T_c for this case.
Hint: You may use that $1 + \frac{1}{2^3} + \frac{1}{3^3} + \ldots = 1.202\ldots$

Problem 12.3 *Bose-Einstein condensation in a gravitational field* (9 pts.)
Consider an ideal Bose gas at temperature T, held in a three-dimensional box of volume $V = L^3$ and subject to a homogeneous gravitational field with acceleration g. Each particle has mass m and an inner degree of freedom ω. The total number of particles is conserved.

(a) Show that the grand partition function is given by

$$\ln Z_G = -\omega \ln(1-\xi) + \omega \frac{V}{\lambda_T^3} \sum_{N=0}^{\infty} \frac{z^{N+1}}{(N+1)^{7/2}} \frac{1-e^{-\beta mgL(N+1)}}{\beta mgL},$$

where $\xi = e^{-\beta mgh} z$.

(b) Bose-Einstein condensation corresponds to $z = 1$. Calculate the particle number N and N/V making use of the approximation

$$g_{5/2}(z) = g_{5/2}(1) - \alpha g_{3/2}(1) + \frac{4}{3}\sqrt{\pi}\alpha^{3/2} + o(\alpha^2)$$

for $z = e^{-\alpha}$ where α is small and positive.

(c) Derive the relation between the critical temperature with and without gravitational field, T_c and T_c^0, respectively. Is the critical temperature smaller or larger than that without the graviational field?