Theoretical Physics VI: Statistical Physics - Theory of Heat

Problem Set 10

due: 9. 1. 2008, 10:15 am

Happy holidays!

Please study the chapter on identical particles and second quantization in the advanced quantum mechanics book of your choice (e.g. Nolting, volume 5/2, chapter 8 or Schwabl, Advanced Quantum Mechanics, chapter 1).

Problem 10.1 *Ising model in the mean field approximation* (15 pts.)
A popular model of ferromagnetism proposed by Ising assigns spins to sites of a lattice. It is given in terms of the Hamiltonian,

$$\hat{\mathsf{H}} = -J \sum_{\langle i,j \rangle=1}^{N} \hat{\mathsf{S}}_i \hat{\mathsf{S}}_j - \mu B \sum_{i=1}^{N} \hat{\mathsf{S}}_i \,,$$

where the first sum is over nearest neighbors only, and J denotes the interaction strength. The spin operators, $\hat{\mathsf{S}}_i$, have eigenvalues ± 1, and B denotes an external magnetic field. In the mean-field approximation, a given spin $\hat{\mathsf{S}}_i$ 'feels' only the interaction with the average of the other spins, $\langle \hat{\mathsf{S}} \rangle$, and the Hamiltonian reduces to

$$\hat{\mathsf{H}} = -\sum_{i=1}^{N} E(J,B) \hat{\mathsf{S}}_i \quad \text{with} \quad E(J,B) = \frac{1}{2} J \nu \langle \hat{\mathsf{S}} \rangle + \mu B \,,$$

and ν the number of nearest neighbors.
(a) Calculate the partition function and the specific Gibbs free energy, $g(J,B)$, and show that

$$\langle \hat{\mathsf{S}} \rangle = \tanh\left(\langle \hat{\mathsf{S}} \rangle \frac{T_c}{T} + \beta \mu B \right) \quad \text{with} \quad T_c = \frac{\nu J}{2 k_B} \,. \tag{1}$$

(b) Interpret Eq. (1) geometrically for $B = 0$.
(c) Show that without any external field, the order parameter $\langle \hat{\mathsf{S}} \rangle$ has the following temperature dependences,

$$\begin{aligned} \langle \hat{\mathsf{S}} \rangle &\approx 1 - \frac{2T}{T_c} e^{-2T_c/T} \quad \text{if} \quad T \sim 0\,\mathrm{K} \,, \\ \langle \hat{\mathsf{S}} \rangle &\approx \sqrt{3(1 - T/T_c)} \quad \text{if} \quad T \sim T_c \,. \end{aligned}$$

(d) Compute the jump in the heat capacity at $T = T_c$.
(e) Calculate the magnetic susceptibility,

$$\chi_{T,N}(B=0) = \left. \frac{\partial M}{\partial B} \right|_{T,N} ,$$

in the neighborhood of $T = T_c$ for both $T > T_c$ and $T < T_c$. What is the critical exponent for both cases?

Problem 10.2 *Correct counting statistics* (3 pts.)
Consider a system of three particles and three single-particle energy levels, $E_i = 0, \epsilon, 2\epsilon$. Calculate the partition function for
(a) bosons, (b) spin-polarized fermions, and (c) distinguishable particles.

Problem 10.3 *Conservation of symmetry* (3 pts.)
Show that the time evolution operator, $\hat{\mathbf{U}}(t, t_0)$, commutes with any permutation operator, $\hat{\mathbf{P}}_{ij}$ for a system of identical particles. The Hamiltonian may be explicitly time-dependent.

Problem 10.4 *Commutators* (4 pts.)
Derive the following commutators for the occupation number operator, creator and annihilator,

$$\left[\hat{\mathbf{n}}_\alpha, \hat{\mathbf{a}}_\beta^+\right]_- \quad , \quad \left[\hat{\mathbf{n}}_\alpha, \hat{\mathbf{a}}_\beta\right]_- \ .$$

Compare the results for bosons and for fermions.