Theoretical Physics VI: Statistical Physics - Theory of Heat

Problem Set 9

due: 19. 12. 2007, 10:15 am

Problem 9.1 *Paramagnetic heat engine* (8 pts.)
A Carnot engine uses a paramagnetic substance as its working substance. The equation of state is

$$M = \frac{nDH}{T}\,,$$

where M is the magnetization, H is the magnetic field, n is the number of moles, D is a constant determined by the type of substance, and T is the temperature.
(a) Show that the internal energy, U, and therefore the heat capacity, c_M, can only depend on the temperature and not on the magnetization. Let us assume that $C_M = C =$constant.
(b) Sketch a typical Carnot cycle in the M-H plane.
(c) Compute the total heat absorbed and the total work done by the Carnot engine.
(d) Compute the efficiency of the Carnot engine.

Problem 9.2 *Phase transition in a van der Waals gas* (11 pts.)
The equation of state for the van der Waals gas is given by

$$\left(p + \frac{a}{v^2}\right)(v-b) = RT\,, \tag{1}$$

where $v = V/N$ denotes the specific volume.
(a) Eq. (1) does not fullfil the thermodynamic stability condition,

$$\left.\frac{\partial p}{\partial v}\right|_T \leq 0$$

for all possible values of v and T. When does Eq. (1) correspond to a stable state, when to an instable one? Indicate the instability region in the $p-V$ diagram.
(b) The critical isoterm is obtained when

$$\left.\frac{\partial p}{\partial v}\right|_T = 0\,.$$

The inflection point of the critical isotherm is called critical point. Calculate the critical pressure, critical specific volume and critical temperature, p_c, v_c, T_c, in terms of the van der Waals parameters a and b and the molar gas constant R.
(c) Convert Eq. (1) to critical units, i.e. substitute $T' = T/T_c$, $v' = v/v_c$, $p' = p/p_c$. Interpret your result.
(d) Derive Maxwell's rule to determine the equilibrium pressure for coexistence of liquid and gaseous phase for a given isotherm from the equilibrium condition for the chemical potentials. Interpret your answer geometrically.
(e) Determine the relative particle numbers $n_l = N_l/N$ and $n_g = N_g/N$ in terms of the specific volumes of liquid and gaseous phase. Interpret your answer geometrically.
(f) Sketch the behavior of the free energy as a function of v for a temperature $T < T_c$.

Indicate the effect of Maxwell's rule on $f(v)$.
optional: Determine the thermodynamic boundaries for the phase transition, v_l, v_g, *from the equilibrium conditions* $\mu_l(p,T) = \mu_g(p,T)$ *and* $p_l(v_l,T) = p_g(v_g,T)$.
Hint: Mathematica or Matlab might be of help.

Problem 9.3 *Thermodynamics of ice skating* (6 pts.)
The pressure of skates onto ice results in a decrease of the freezing point of water and in a melting of ice.
(a) Assume the mass of the skater to be 80 kg. The blade is touching the ice on a length of 20 cm and a width of 4 mm. The specific volumes of water and ice are $1.0 \cdot 10^{-3}\,\mathrm{m}^3/\mathrm{kg}$ and $1.1 \cdot 10^{-3}\,\mathrm{m}^3/\mathrm{kg}$, respectively. The latent heat of ice is $3.4 \cdot 10^{-5}\,\mathrm{J/kg}$. Calculate the decrease of the freezing point in the static case. Estimate whether this effect is sufficient to produce a film of water on which the skates can glide.
(b) Calculate the vapor pressure curve, $p = p(T)$, from the Clausius-Clapeyron equation. You may assume that (i) $v_{liquid} \ll v_{gas}$ and that (ii) the gas phase obeys the equation of state for the ideal gas. The initial values are p_0 and T_0 and the vaporization heat does not depend on temperature.