Theoretical Physics VI: Statistical Physics - Theory of Heat

Problem Set 7

due: 5. 12. 2007, 10:15 am

Problem 7.1 *Einstein theory of the heat capacity in solids* (5 pts.)
Consider a system of non-interacting harmonic oscillators on the sites of a lattice of N atoms. All oscillators have the same frequency ω and the energy of this system is given by

$$E = \sum_{i}^{3N} \hbar\omega n_i + \frac{3N}{2}\hbar\omega$$

with occupation numbers n_i.
(a) As a prerequisite, show in how many different ways 4 indistinguishable balls ($\hat{=}$ quanta) can be distributed into 3 pots ($\hat{=}$ lattice sites). Then calculate the number of states with energy E, $\Omega(E)$.
(b) Calculate the entropy, the energy and the heat capacity of the system. Compare your results to those obtained in the lecture.

Problem 7.2 *Thermal noise in a circuit* (4 pts.)
Some passive electronic components (resistors, capacitors, inductors, transformers, etc.) are assembled to a circuit. Assume that the circuit contains no sources, and that it is held at temperature T.
(a) Find the probability density, $p(v)$, that a voltage, v, will exist on a capacitor of capacitance, C. What is the root mean square voltage, $\sqrt{\langle v^2 \rangle}$, given that $T = 300\,\mathrm{K}$ and $C = 100\,\mathrm{pF}$?
(b) Find the probability density, $p(i)$, for the current, i, through an inductor of inductance L. What is the root mean square current, $\sqrt{\langle i^2 \rangle}$, given that $T = 300\,\mathrm{K}$ and $L = 1\,\mathrm{mH}$?
Hint: Consider the energy stored in the capacitor, or inductor, and treat the respective component as system of interest which is immersed in a temperature reservoir (the circuit).
optional: Why does this method not work for the voltage on a resistor?

Problem 7.3 *Cooling and trapping of neutral atoms* (6 pts.)
A gas of N non-interacting atoms is trapped with the potential of the trap given by

$$V(r) = ar \qquad \text{with} \qquad r = \sqrt{x^2 + y^2 + z^2}\,.$$

The gas is in thermal equilibrium, and the temperature T is high enough such that the atoms can be treated classically.
(a) Find the single partition function Z_1 for a trapped atom. State your answer in the form $Z_1 = AT^{\alpha}a^{-\eta}$. What are A, α and η?
(b) Find the entropy of the gas in terms of N and $Z_1 = Z_1(T, a)$.
(c) The atoms can be cooled if the potential is lowered reversibly by decreasing a while no heat exchange with the surroundings is allowed. Find T as a function of a and the initial values T_0 and a_0.

Problem 7.4 *Simple model of the heat capacity in glasses* (6 pts.)

In a simple model of glasses, the particles are subject to a double well potential. At low temperatures, the double well potential can be approximated by a two-level system. Consider a system of N non-interacting particles at temperature T where particle i is in a state with energy $\pm\Delta_i$.

(a) In a first step, assume all Δ_i to be equal, i.e. $\Delta_i = \Delta$. Calculate for one particle the partition function and the mean energy, and derive the specific heat.

Note: The specific heat, C_V, is related to the heat capacity, c_V,

$$c_V = \frac{C_V}{N} = \frac{\partial \langle E \rangle}{\partial T} \, .$$

(b) Now assume the available values Δ_i to be evenly distributed in an interval $[0, \Delta_0]$. Show that $C_V \propto T$ for $k_B T \ll \Delta_0$.

Hint: Consider the mean contribution of particle i to C_V. You may assume $k_B T \ll \Delta_i$, since it is the states with energies $\pm\Delta_i \gg k_B T$ which yield the largest contribution in the regime $k_B T \ll \Delta_0$. C_V can then be obtained by summing up the mean contributions and replacing the sum by an integral.

Problem 7.5 *Exact differentials* (4 pts.)

Which of the following differentials is exact? For those cases, in which the differential is exact, find the function $u(x, y)$.

$$\begin{aligned}
&\text{(a)} \qquad du &=& \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2} \, , \\
&\text{(b)} \qquad du &=& \left(y - x^2\right) dx + \left(x + y^2\right) dy \, , \\
&\text{(c)} \qquad du &=& \left(2y^2 - 3x\right) dx + 4\,x\,y\,dy \, .
\end{aligned}$$