Theoretical Physics VI: Statistical Physics - Theory of Heat

Problem Set 4

due: 14. 11. 2007, 10:15 am

Problem 4.1 *Spin lattice* (6 pts.)
Consider an isolated crystal lattice. An unpaired electron with spin 1/2 is localized at each site of the lattice. An external magnetic field $\vec{B} = B\vec{e}_z$ is applied, and the spins can point either parallel or anti-parallel to the magnetic field ($s_{z,i} = \pm 1/2$). Therefore the states describing the system are of the form

$$r = (s_{z,1}, s_{z,2}, \ldots, s_{z,N}) \,,$$

where N denotes the total number of lattices sites and hence the total number of spins. The energy of state r is given by

$$E_r(B) = -2\mu_B B \sum_{i=1}^{N} s_{z,i} \,.$$

(a) Calculate the energy E_n for n spins pointing parallel to the magnetic field. What is the number of states $\Omega_n(E_n)$ with that energy?
(b) Apply Stirling's formula

$$M! \approx (M/e)^M \qquad (M \gg 1)$$

to $\Omega_n(E_n)$. You may assume $N - n \gg 1$ and $n \gg 1$. Expressing n by E (cf. part a), show that

$$\ln \Omega(E, B, N) = -\frac{N}{2}\left(1 - \frac{E}{N\mu_B B}\right) \ln\left(\frac{1}{2} - \frac{E}{2N\mu_B B}\right) - \frac{N}{2}\left(1 + \frac{E}{N\mu_B B}\right) \ln\left(\frac{1}{2} + \frac{E}{2N\mu_B B}\right) \,.$$

Problem 4.2 *Mixture of ideal gases* (4 pts.)
A vessel of volume V contains a mixture of ideal gases with N_i particles of the kind i, $i = 1, \ldots, m$. Calculate the total entropy $S(E, V, N_1, \ldots, N_m)$ and the pressure.

Problem 4.3 *Ideal gas in a centrifuge* (9 pts.)
A cylindrical centrifuge with radius R and height L contains an ideal gas. The centrifuge rotates at angular frequency ω. Choosing the z-axis along the axis of the centrifuge, the dynamics of one particle is described by the Hamiltonian function

$$H = \frac{p^2}{2m} - \omega(xp_y - yp_x) \,,$$

where m is the mass of the particles.
(a) Show that the partition function Z_1 for one particle of the gas is given by

$$Z_1 = \left(\frac{2\pi m}{h^2\beta}\right)^{3/2} \frac{2\pi L}{m\beta\omega^2}\left(e^{m\beta\omega^2 R^2/2} - 1\right) \,.$$

Hint: First integrate over momentum, then employ cylindrical coordinates for the spatial integrals.
(b) Calculate the partition function Z if there are N identical particles in the cylinder.
(c) Calculate the Helmholtz free energy F of the gas in the centrifuge.
(d) Calculate the pressure and the total force on the outer wall of the centrifuge.

Problem 4.4 *Doppler broadening* (6 pts.)
Consider a container with a gas of atoms, each of mass m, at temperature T. The atoms emit light which passes in the x-direction through a window and hits a spectrometer. A stationary atom would emit light at the frequency ν_0. But, because of the Doppler effect, the frequency observed from an atom moving with velocity v_x is given by

$$\nu = \nu_0 \left(1 + \frac{v_x}{c}\right) ,$$

where c is the speed of light. Therefore the light arriving at the spectrometer is characterized by an intensity distribution $I(\nu)$. Calculate
(a) the mean frequency $\langle \nu \rangle$ of the observed light,
(b) the root mean square frequency shift $\Delta\nu = \sqrt{\langle (\nu - \langle \nu \rangle)^2 \rangle}$,
(c) the intensity distribution $I(\nu)$ in general and for the non-relativistic case ($kT \ll mc^2$).