Theoretical Physics VI: Statistical Physics - Theory of Heat

Problem Set 3

due: 7. 11. 2007, 10:15 am

Problem 3.1 *Discrete random variable* (4 pts.)
The x component of the angular momentum of some quantum system can only take on three values: $-\hbar$, 0, or $\hbar$. For a given state of the system, it is known that

$$\langle L_x \rangle = \frac{\hbar}{3} \quad \text{and} \quad \langle L_x^2 \rangle = \frac{2\hbar^2}{3} \, .$$

(a) What is the probability density $p(L_x)$ for the x component of the angular momentum? Sketch $p(x)$.
(b) Sketch also the cumulative function $F(L_x)$.

Problem 3.2 *Classical particles in a box* (6 pts.)
(a) Consider a classical particle of mass m moving in one dimension, and denote its coordinate and momentum by x and p, respectively. The particle is confined to a box so that it is located between $x = 0$ and $x = L$. Suppose the energy of the particle is know to lie between E and $E + \delta E$. Draw the classical phase space of this particle, indicating which regions are accessible to the particle.
(b) Now the same box contains two weakly interacting classical particles of mass m with positions x_1, x_2 and momenta p_1, p_2. The total energy of the system is known to lie between E and $E + \delta E$. Since it is difficult to draw in a four-dimensional phase space, sketch separately the part of phase space involving x_1 and x_2, and that involving p_1 and p_2. Indicate the regions of phase space accessible to the system.

Problem 3.3 *Non-interacting fermions* (6 pts.)
Two non-interacting spinless fermions in a one-dimensional harmonic trap have the joint probability density $p(x_1, x_2)$ for finding particle 1 at $x = x_1$ and particle 2 at $x = x_2$,

$$p(x_1, x_2) = \frac{1}{\pi x_0^2} \left(\frac{x_2 - x_1}{x_0} \right)^2 \exp\left[-\frac{x_1^2 + x_2^2}{x_0^2} \right] \qquad (1)$$

with x_0 denoting a characteristic distance.

Note: Except for the optional question, no explicit knowledge of quantum mechanics is required here.

(a) On a sketch of the x_1, x_2-plane, indicate where the joint probability is highest and where it is minimum.
(b) Find the probability for finding particle 1 at the position x. Do the same for particle 2. Are x_1 and x_2 statistically independent?
(c) Find the conditional probability density $p(x_1|x_2)$ that particle 1 is at $x = x_1$ given that particle 2 is known to be at $x = x_2$. Sketch the result.
optional: For non-interacting spinless bosons, the term $x_1 - x_2$ in Eq. (1) should be replaced by $x_1 + x_2$. Why? How does this affect $p(x_1|x_2)$?

Problem 3.4 *Waiting times* (9 pts.)

The average number of cars and buses passing by an observer on a one-way road are equal: Each hour there are 12 buses and 12 cars passing by on average. While the buses are scheduled, i.e. each bus appears exactly 5 minutes after the previous one, the cars appear at random. The probability that a car passes in a time interval dt is given by dt/τ with $\tau = 5$ minutes.

(a) Verify that the average number of cars passing in one hour is indeed 12.

(b) In a randomly chosen 10 minute interval, what are the probabilities $P_{bus}(n)$ and $P_{car}(n)$ that n buses and n cars pass by? Calculate also the mean and the variance of these probabilities.

(c) What are the probability distributions $P_{bus}(\Delta t)$ and $P_{car}(\Delta t)$ for the time interval Δt between two successive buses and cars, respectively? What are the mean and the variance of these distributions?

(d) What is the probability distribution of the time which another observer who arrives at the road at a randomly chosen time, has to wait for the first bus to arrive? What is the probability distribution of the time this person has to wait for the first car to pass by? What are the mean and the variance of these distributions? How does your answer compare to (c)?

optional: The same concepts can be applied to the motion of atoms in a dilute gas (without any external fields): The atoms move in straight lines, and at certain times they collide with each other. Which scenario – collisions at equally spaced times or collisions at random times – describes the atoms in a dilute gas better?