Theoretical Physics VI: Statistical Physics - Theory of Heat

Problem Set 2

due: 31. 10. 2007, 10:15 am

Problem 2.1 *Non-interacting spins* (6 pts.)
A system of N spins without any external field or interaction between the spins has equal probability for a single spin to be up or down.
(a) Write down the probability $p_N(m)$ of having m spins up and $N-m$ down. Show that

$$\sum_{m=0}^{N} p_N(m) = 1\,.$$

(b) Calculate the mean $\langle m \rangle$ and the variance $\sqrt{\langle \Delta m^2 \rangle}$. The dimensionless magnetization is defined by $M = 2m - N$. Calculate its mean and variance.

Problem 2.2 (3 pts.)
Two editors proof-read a book. Editor A finds 200 typos while editor B finds only 150 typos. Of all the typos found by editors A and B, 100 were found by both editors. How many typos were not discovered?

Problem 2.3 *Probability & entropy* (8 pts.)
The entropy of a probability distribution P is defined according to $S = -k \sum P \ln P$. Given that the joint probability of two events is $P(A \cap B) = P(B|A) \cdot P(A)$ or

$$P^{(A,B)}(i,j) = P^{(B)}(j|i) \cdot P^{(A)}(i)\,, \quad i \in A,\, j \in B\,,$$

calculate the entropy $S = S_{(A,B)}$ of the distribution $P = P^{(A,B)}(i,j)$
(a) assuming strongly correlated events, i.e.

$$P^{(B)}(j|i) = \delta_{ij} \quad i,j = 1,2,\ldots,n\,,$$

(b) assuming statistically independent events.
(c) Interpret these results.
(d) Show that the entropy S of the distribution $P = P(i)$ is maximum for a uniform distribution, i.e. if $P(i) = 1/N$, then $S = k \ln N$.

Problem 2.4 *Continuous random variable* (8 pts.)
Consider a particle performing harmonic motion, $x(t) = x_0 \sin(\omega t + \phi)$ where the phase ϕ is unknown. The amount of time the particle spends between x and $x + dx$ is inversely proportional to its speed (i.e. the magnitude of its velocity) at x. Thinking in terms of an ensemble of similarly prepared oscillators, the probability density $p(x)$ for finding an oscillator at x is proportional to the time this oscillator spends near x.
(a) Find the speed at x as a function of x, ω and x_0.
(b) Find $p(x)$.
Hint: Use normalization to find the proportionality constant.
(c) Sketch $p(x)$. What are the most probable, the least probable and the mean value of $p(x)$?