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3 Übungsblatt Theoretische Physik V

3.1 (Lorentz-Gruppe)

Die Basisgeneratoren $L^{\alpha\beta}$ der Lorentz-Gruppe sind im Minkowski-Raum durch die Darstellungsmatrizen

$$\left(L^{\alpha\beta}\right)^{\mu}{}_{\nu} = i\left(g^{\alpha\mu}g^{\beta}{}_{\nu} - g^{\beta\mu}g^{\alpha}{}_{\nu}\right) \tag{1}$$

gegeben.

a)

Es ist zu zeigen, dass die Symmetrieeigenschaften gelten:

$$\left(L^{\alpha\beta}\right)^{\mu}{}_{\nu} + \left(L^{\alpha\beta}\right)_{\nu}{}^{\mu} = 0 \tag{2}$$

$$\left(L^{\alpha\beta}\right)^{\mu}{}_{\nu} + \left(L^{\beta\alpha}\right)^{\mu}{}_{\nu} = 0 \tag{3}$$

Wir überprüfen durch einsetzen von (1) in (2) bzw. (3):

$$\begin{aligned}(2):\ i\left(g^{\alpha\mu}g^{\beta}{}_{\nu} - g^{\beta\mu}g^{\alpha}{}_{\nu}\right) + i\left(g^{\alpha}{}_{\nu}g^{\beta\mu} - g^{\beta}{}_{\nu}g^{\alpha\mu}\right) &= 0\\ \underbrace{\left(g^{\alpha\mu}g^{\beta}{}_{\nu} - g^{\beta}{}_{\nu}g^{\alpha\mu}\right)}_{=0} + \underbrace{\left(g^{\alpha}{}_{\nu}g^{\beta\mu} - g^{\beta\mu}g^{\alpha}{}_{\nu}\right)}_{=0} &= 0\end{aligned}$$

und

$$\begin{aligned}(3):\ i\left(g^{\alpha\mu}g^{\beta}{}_{\nu} - g^{\beta\mu}g^{\alpha}{}_{\nu}\right) + i\left(g^{\beta\mu}g^{\alpha}{}_{\nu} - g^{\alpha\mu}g^{\beta}{}_{\nu}\right) &= 0\\ \underbrace{\left(g^{\alpha\mu}g^{\beta}{}_{\nu} - g^{\alpha\mu}g^{\beta}{}_{\nu}\right)}_{=0} + \underbrace{\left(g^{\beta\mu}g^{\alpha}{}_{\nu} - g^{\beta\mu}g^{\alpha}{}_{\nu}\right)}_{=0} &= 0.\end{aligned}$$

□

und der Kommutator der Basisgeneratoren zu verifizieren:

$$\left[L^{\alpha\beta}, L^{\gamma\delta}\right] = i\left(g^{\alpha\delta}L^{\beta\gamma} + g^{\beta\gamma}L^{\alpha\delta} - g^{\alpha\gamma}L^{\beta\delta} - g^{\beta\delta}L^{\alpha\gamma}\right). \tag{4}$$

Wir berechnen den Kommutator:

$$\left[L^{\alpha\beta}, L^{\gamma\delta}\right] = L^{\alpha\beta}L^{\gamma\delta} - L^{\gamma\delta}L^{\alpha\beta}.$$

Wir betrachten das Element $\mu\nu$ des Kommutators:

$$\begin{aligned}
\left[L^{\alpha\beta}, L^{\gamma\delta}\right]^{\mu}_{\ \nu} &= \left(L^{\alpha\beta}\right)^{\mu\lambda}\left(L^{\gamma\delta}\right)_{\lambda\nu} - \left(L^{\gamma\delta}\right)^{\mu\lambda}\left(L^{\alpha\beta}\right)_{\lambda\nu} \\
&= (ii)\left(g^{a\mu}g^{\beta\lambda} - g^{\beta\mu}g^{\alpha\lambda}\right)\left(g^{\gamma}_{\lambda}g^{\delta}_{\nu} - g^{\delta}_{\lambda}g^{\gamma}_{\nu}\right) \\
&\quad - (ii)\left(g^{\gamma\mu}g^{\delta\lambda} - g^{\delta\mu}g^{\gamma\lambda}\right)\left(g^{\alpha}_{\lambda}g^{\beta}_{\nu} - g^{\beta}_{\lambda}g^{\alpha}_{\nu}\right) \\
&= -\left(g^{a\mu}g^{\beta\gamma}g^{\delta}_{\nu} - g^{\alpha\mu}g^{\beta\delta}g^{\gamma}_{\nu} - g^{\beta\mu}g^{\alpha\gamma}g^{\delta}_{\nu} + g^{\beta\mu}g^{\alpha\delta}g^{\gamma}_{\nu}\right) \\
&\quad + \left(g^{\gamma\mu}g^{\delta\alpha}g^{\beta}_{\nu} - g^{\gamma\mu}g^{\delta\beta}g^{\alpha}_{\nu} - g^{\delta\mu}g^{\gamma\alpha}g^{\beta}_{\nu} + g^{\delta\mu}g^{\gamma\beta}g^{\alpha}_{\nu}\right) \\
&= g^{\beta\gamma}\left(g^{\delta\mu}g^{\alpha}_{\nu} - g^{\alpha\mu}g^{\delta}_{\nu}\right) - g^{\beta\delta}\left(g^{\gamma\mu}g^{\alpha}_{\nu} - g^{\alpha\mu}g^{\gamma}_{\nu}\right) \\
&\quad - g^{\alpha\gamma}\left(g^{\delta\mu}g^{\beta}_{\nu} - g^{\beta\mu}g^{\delta}_{\nu}\right) + g^{\alpha\delta}\left(g^{\beta\mu}g^{\gamma}_{\nu} - g^{\gamma\mu}g^{\beta}_{\nu}\right) \\
&= i\left(g^{\beta\gamma}L^{\alpha\delta} - g^{\beta\delta}L^{\alpha\gamma} - g^{\alpha\gamma}L^{\beta\delta} + g^{\alpha\delta}L^{\beta\gamma}\right)^{\mu}_{\ \nu}
\end{aligned}$$

□

b)

Die Basisgeneratoren können in zwei Klassen eingeteilt werden:

$$M_k = L^{0k} \tag{5}$$

$$L_k = \frac{1}{2}\varepsilon_{klm}L^{lm} \tag{6}$$

Es sind folgende Kommutatoren zu verifizieren:

$$[L_i, L_j] = i\varepsilon_{ijk}L_k \tag{7}$$

$$[L_i, M_j] = i\varepsilon_{ijk}M_k \tag{8}$$

$$[M_i, M_j] = -i\varepsilon_{ijk}L_k \tag{9}$$

Wir setzen (6) in (7) ein und erhalten:

$$[L_i, L_j] = \frac{1}{4}\left[\varepsilon_{i\alpha\beta}L^{\alpha\beta}, \varepsilon_{j\gamma\delta}L^{\gamma\delta}\right] = \frac{1}{4}\varepsilon_{i\alpha\beta}\varepsilon_{j\gamma\delta}\left[L^{\alpha\beta}L^{\gamma\delta} - L^{\gamma\delta}L^{\alpha\beta}\right] = \frac{1}{4}\varepsilon_{i\alpha\beta}\varepsilon_{j\gamma\delta}\left[L^{\alpha\beta}, L^{\gamma\delta}\right]$$

Einsetzen von (4) liefert:

$$\begin{aligned}
[L_i, L_j] &= \frac{1}{2}i\varepsilon_{i\alpha\beta}\frac{1}{2}\varepsilon_{j\gamma\delta}\left(g^{\alpha\delta}L^{\beta\gamma} + g^{\beta\gamma}L^{\alpha\delta} - g^{\alpha\gamma}L^{\beta\delta} - g^{\beta\delta}L^{\alpha\gamma}\right) \\
&= i\varepsilon_{ijk}L_k
\end{aligned}$$

Wir definieren folgende Darstellungen

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \qquad L_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

und rechnen damit nach

$$\begin{aligned}
[L_1, L_2] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = iL_3
\end{aligned}$$

$$\begin{aligned}
[L_2, L_3] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\
&= i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = iL_1
\end{aligned}$$

$$[L_3, L_1] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} = iL_2$$

$$[L_1, M_2] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = iM_3$$

$$[L_2, M_3] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = iM_1$$

$$[L_3, M_1] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = iM_2$$

$$[M_1, M_2] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$
= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -iL_3
$$

$$
\begin{aligned}
[M_2, M_3] &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = -iL_1
\end{aligned}
$$

$$
\begin{aligned}
[M_3, M_1] &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = -iL_2
\end{aligned}
$$