Übungen zur Theoretischen Physik III, Elektrodynamik - Blatt 1

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1 Vektoranalysis

Es gilt für die Aufgabe: $\mathsf{c} = (c_x, c_y, c_z)$ ein konstanter Vektor, $\mathrm{x} = (x, y, z)$ und $\nabla = (\partial_x, \partial_y, \partial_z)$

1.1 a) $\nabla(\mathsf{c} \cdot \mathrm{x}) = \nabla(c_x x + c_y y + c_z z) = (c_x, c_y, c_z) = \mathsf{c}$

1.2 b) $\nabla \cdot \mathrm{x} = (\partial_x x + \partial_y y + \partial_z z) = (1 + 1 + 1) = 3$

1.3 c) $\nabla \times \mathrm{x} = (\partial_y z - \partial_z y)\overrightarrow{e_x} + (\partial_z x - \partial_x z)\overrightarrow{e_y} + (\partial_x y - \partial_y x)\overrightarrow{e_z} = 0$

1.4 d) $\nabla \times (\mathsf{c} \times \mathrm{x}) = \nabla \times (\epsilon_{\alpha\beta\gamma} c_\alpha x_\beta \overrightarrow{e_\gamma}) = \partial_\delta \varepsilon_{\alpha\beta\gamma} c_\alpha x_\beta \varepsilon_{\delta\gamma\sigma} \overrightarrow{e_\sigma}$

$$\Leftrightarrow = \partial_\delta c_\alpha x_\beta \varepsilon_{\alpha\beta\gamma} \varepsilon_{\gamma\sigma\delta} \overrightarrow{e_\sigma} = \partial_\delta c_\alpha x_\beta \overrightarrow{e_\sigma}(\delta_{\alpha\sigma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\sigma}) = c_\alpha \overrightarrow{e_\alpha} \partial_\beta x_\beta - \partial_\alpha c_\alpha x_\beta \overrightarrow{e_\beta}$$

$$\Leftrightarrow \nabla \times (\mathsf{c} \times \mathrm{x}) = 3\mathsf{c} - \mathsf{c} = 2\mathsf{c}$$

wobei wir $\varepsilon_{\alpha\beta\gamma}\varepsilon_{\gamma\sigma\delta} = \delta_{\alpha\sigma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\sigma}$, $(\overrightarrow{e_\alpha} \times \overrightarrow{e_\beta}) = \varepsilon_{\alpha\beta\gamma}\overrightarrow{e_\gamma}$ und die Einsteinsche Summenkonvention genutzt haben, wobei $\overrightarrow{e_i}$ mit $i = x, y, z, \alpha, \beta, \gamma, \ldots$.

2 Vektorfeld

Es gilt für die Aufgabe: $\mathsf{d} = (d_x, d_y, d_z)$ ein konstanter Vektor, $r = |\mathrm{x}| = \sqrt{x^2 + y^2 + z^2}$ und $\mathrm{A} = \mathsf{d} \times \frac{x}{r^3}$.

2.1 a) $\overrightarrow{E}(\mathrm{x}) = -\nabla(\mathsf{d} \cdot \frac{\mathrm{x}}{r^3}) = -\nabla(\frac{d_x x + d_y y + d_z z}{r^3})$

$$\Leftrightarrow = -(\partial_x \frac{d_x x + d_y y + d_z z}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \partial_y \frac{d_x x + d_y y + d_z z}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \partial_z \frac{d_x x + d_y y + d_z z}{(x^2+y^2+z^2)^{\frac{3}{2}}})$$

$$\Leftrightarrow \overrightarrow{E}(\mathrm{x}) = (\frac{3x}{r^5}(d_x x + d_y y + d_z z) - \frac{d_x}{r^3}, \frac{3y}{r^5}(d_x x + d_y y + d_z z) - \frac{d_y}{r^3}, \frac{3z}{r^5}(d_x x + d_y y + d_z z) - \frac{d_z}{r^3})$$

2.2 b) $\overrightarrow{E}(\mathrm{x}) = \nabla \times \mathrm{A} = \nabla \times (\mathsf{d} \times \frac{\mathrm{x}}{r^3}) = \nabla \times (\varepsilon_{\alpha\beta\gamma} d_\alpha \frac{x_\beta}{r^3} \overrightarrow{e_\gamma})$

$$\Leftrightarrow = \partial_\delta \varepsilon_{\alpha\beta\gamma} d_\alpha \frac{x_\beta}{r^3} \varepsilon_{\delta\gamma\sigma} \overrightarrow{e_\sigma} = \partial_\delta d_\alpha \frac{x_\beta}{r^3} \varepsilon_{\alpha\beta\gamma} \varepsilon_{\gamma\sigma\delta} \overrightarrow{e_\sigma}$$

$$\Leftrightarrow = \partial_\delta d_\alpha \frac{x_\beta}{r^3} \overrightarrow{e_\sigma} (\delta_{\alpha\sigma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\delta})$$

$$\Leftrightarrow = \partial_\beta \frac{x_\beta}{r^3} d_\alpha \overrightarrow{e_\alpha} - \partial_\alpha \frac{d_\alpha}{r^3} x_\beta \overrightarrow{e_\beta}$$

$$\Leftrightarrow = (\frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5})\mathsf{d} - (\partial_x \frac{d_x}{r^3} x + \partial_y \frac{d_y}{r^3} x + \partial_z \frac{d_z}{r^3} x, \ldots, \ldots)$$

$$\Leftrightarrow = (\frac{3}{r^3} - \frac{3r^2}{r^5})\mathsf{d} + \frac{3}{\mathrm{r}^5}(\mathrm{d_x x + d_y y + d_z z})\mathrm{x} - \frac{\mathsf{d}}{r^3}$$

$$\Leftrightarrow = 0\mathsf{d} + \frac{3}{\mathrm{r}^5}(\mathrm{d_x x + d_y y + d_z z})\mathrm{x} - \frac{\mathsf{d}}{r^3}$$

$$\overrightarrow{E}(\mathrm{x}) = (\frac{3x}{r^5}(d_x x + d_y y + d_z z) - \frac{d_x}{r^3}, \frac{3y}{r^5}(d_x x + d_y y + d_z z) - \frac{d_y}{r^3}, \frac{3z}{r^5}(d_x x + d_y y + d_z z) - \frac{d_z}{r^3})$$

Der Vergleich der Ergebnisse von 2a) und 2b) zeigt die Identität der Schreibweisen.

2.3 c) Divergenz von E : $\nabla \cdot \overrightarrow{E}(\mathrm{x}) = \nabla \cdot -\nabla(\mathsf{d} \cdot \frac{\mathrm{x}}{r^3}) = -\Delta(\mathsf{d} \cdot \frac{\mathrm{x}}{r^3})$

$$\Leftrightarrow = \nabla \cdot (\frac{3x}{r^5}(d_x x + d_y y + d_z z) - \frac{d_x}{r^3}, \frac{3y}{r^5}(d_x x + d_y y + d_z z) - \frac{d_y}{r^3}, \frac{3z}{r^5}(d_x x + d_y y + d_z z) - \frac{d_z}{r^3})$$

$$\Leftrightarrow = \partial_x \{\frac{3x}{r^5}(d_x x + d_y y + d_z z) - \frac{d_x}{r^3}\} + \partial_y \{\frac{3y}{r^5}(d_x x + d_y y + d_z z) - \frac{d_y}{r^3}\} + \partial_z \{\frac{3z}{r^5}(d_x x + d_y y + d_z z) - \frac{d_z}{r^3}\}$$

$$\Leftrightarrow = \frac{3}{r^5}(3d_x x + d_y y + d_z z) - \frac{15x^2}{r^7}(d_x x + d_y y + d_z z) + \frac{3}{r^5}(d_x x + 3d_y y + d_z z) - \ldots$$

$$\ldots \frac{15y^2}{r^7}(d_x x + d_y y + d_z z) + \frac{3}{r^5}(d_x x + d_y y + 3d_z z) - \frac{15z^2}{r^7}(d_x x + d_y y + d_z z)$$

$$\Leftrightarrow = \frac{3}{r^5}(5d_x x + 5d_y y + 5d_z z) - \frac{15(x^2+y^2+z^2)}{r^7}(d_x x + d_y y + d_z z)$$

$$\Leftrightarrow = (d_x x + d_y y + d_z z)(\frac{15}{r^5} - \frac{15r^2}{r^7})$$

$$\nabla \cdot \overrightarrow{E} = -\Delta \mathrm{A} = (d_x x + d_y y + d_z z)0 = 0$$

Rotation von E :$\nabla \times \overrightarrow{E} = \nabla \times (\nabla \times \mathrm{A})$

$$\Leftrightarrow = \nabla \times (\varepsilon_{\alpha\beta\gamma}\partial_\alpha A_\beta \overrightarrow{e_\gamma}) = \partial_\delta \varepsilon_{\alpha\beta\gamma}\partial_\alpha A_\beta \varepsilon_{\gamma\sigma\delta}\overrightarrow{e_\sigma}$$

$$\Leftrightarrow = \partial_\delta \partial_\alpha A_\beta \overrightarrow{e_\sigma}(\delta_{\alpha\sigma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\sigma}) = \partial_\alpha \overrightarrow{e_\alpha}\partial_\beta A_\beta - \partial_\alpha^2 A_\beta \overrightarrow{e_\beta}$$

$$\Leftrightarrow = 0\partial_\beta A_\beta - \Delta \mathrm{A}$$

$$\Leftrightarrow = \nabla \times \overrightarrow{E} = -\Delta \mathrm{A} = 0$$

Das $-\Delta \mathrm{A} = 0$ gilt kann man aus 2c) ablesen.

3 Kugelsymmetrisches Feld

Es gilt $r = |\mathrm{x}| = \sqrt{x^2 + y^2 + z^2}$ mit $\mathrm{x} = (x, y, z)$.

3.1 a) $\frac{\partial}{\partial x} f(r)$ und $\frac{\partial^2}{\partial x^2} f(r)$

$$\frac{\partial}{\partial x} f(r) = \frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial x} = f'(r)\frac{x}{r}$$

$$\frac{\partial^2}{\partial x^2} f(r) = \frac{\partial}{\partial x}(f'(r)\frac{x}{r}) = \frac{x}{r}\frac{\partial}{\partial x}f'(r) + f'(r)\frac{\partial}{\partial x}\frac{x}{r} = \frac{x}{r}\frac{\partial f'}{\partial r}\frac{\partial r}{\partial x} + f'(r)(\frac{1}{r} - \frac{1}{2}\frac{2x^2}{r^3})$$

$$\Leftrightarrow = f''(r)\frac{x^2}{r^2} + f'(r)(\frac{1}{r} - \frac{x^2}{r^3})$$

$$\Leftrightarrow \frac{\partial^2}{\partial x^2} f(r) = f''(r)\frac{x^2}{r^2} + f'(r)(\frac{r^2 - x^2}{r^3})$$

3.2 b) Einheitsvektor und Betrag von $\nabla f(r)$

$$\nabla f(r) = (\partial_x f(r), \partial_y f(r), \partial_z f(r))$$

$$\Leftrightarrow = (f'(r)\frac{x}{r}, f'(r)\frac{y}{r}, f'(r)\frac{z}{r})$$

$$\Leftrightarrow \nabla f(r) = f'(r)\frac{\mathrm{x}}{r}$$

Für den Betrag von des Feldgradienten gilt :

$$|\nabla f(r)| = |f'(r)\frac{\mathrm{x}}{r}| = |f'(r)|\frac{|\mathrm{x}|}{r} = |f'(r)| = \pm f'(r) = sign(f'(r))$$

Für den Einheitsvektor ϵ folgt also:

$$\epsilon = \frac{f'(r)\frac{\mathrm{x}}{r}}{|f'(r)\frac{\mathrm{x}}{r}|} = \pm\frac{\mathrm{x}}{r}$$