Solid State Theory
Exercise 9

Landau Fermi liquid theory

Exercise 9.1 Uniaxial Compressibility

We consider a system of electrons upon which an uniaxial pressure in z-direction acts. Assume that this pressure causes a deformation of the Fermi surface $k \equiv k_F^0$ of the form

$$k_F(\phi, \theta) = k_F^0 + \gamma \frac{1}{k_F^0} \left[3k_z^2 - (k_F^0)^2 \right] = k_F^0 + \gamma k_F^0 [3 \cos^2 \theta - 1], \tag{1}$$

where $\gamma = (P_z - P_0)/P_0$ is the anisotropy of the applied pressure.

a) Show that for small $\gamma \ll 1$, the deformed Fermi surface $k_F(\phi, \theta)$ encloses the same volume as the non-deformed one, k_F^0, where terms of order $\mathcal{O}(\gamma^2)$ can be neglected.

b) The deformation of the Fermi surface effects a change in the distribution function of the electrons. Using Landau's Fermi Liquid theory, calculate the uniaxial compressibility

$$\kappa_u = \frac{1}{V} \frac{\partial^2 E}{\partial P_z^2}, \tag{2}$$

which is caused by he deformation given in eq. (1) (E denotes the Landau energy functional).

c) What is the stability condition of the Fermi liquid against the deformation given in eq. (1)?

Exercise 9.2 Pomeranchuk instability

It can be shown in general [1] that the Fermi liquid is stable against an arbitrary deformation,

$$k_F(\phi, \theta) = k_F^0 + u_\sigma(\phi, \theta), \tag{3}$$

of the Fermi surface if

$$F_l^s > -(2l+1) \tag{4}$$
$$F_l^a > -(2l+1). \tag{5}$$

Verify this result by considering Landau's energy functional and expanding the displacement $u_\sigma(\phi, \theta)$ in terms of spherical harmonics.

Exercise 9.3 Polarization of a neutral Fermi liquid

Consider a system of neutral spin-1/2 particles each carrying a magnetic moment $\vec{\mu} = \frac{\mu}{2}\vec{\sigma}$. An electric field $\vec{E}$ couples to the atoms by the relativistic spin-orbit interaction

$$H_{SO} = \frac{\mu}{2} \left(\frac{\vec{v}}{c} \times \vec{E} \right) \cdot \vec{\sigma} \tag{6}$$

where $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ is the vector of Pauli spin matrices. In the following we want to calculate the linear response function χ for the uniform polarization

$$\vec{P} = \chi \vec{E}. \tag{7}$$

In the presence of spin-orbit interaction we have to consider a more general situation of a distribution of quasiparticles with variable spin quantization axis. In such a case we must treat the quasiparticle distribution function and the energy as a 2×2 matrix, $(\hat{n}_{\vec{p}})_{\alpha\beta}$ and $(\hat{\epsilon}_{\vec{p}})_{\alpha\beta}$, respectively. Furthermore, we require f to be a scalar under spin rotations. In this case f must be of the form

$$\hat{f}_{\alpha\beta,\alpha'\beta'}(\vec{p},\vec{p}') = f^s(\vec{p},\vec{p}')\delta_{\alpha\beta}\delta_{\alpha'\beta'} + f^a(\vec{p},\vec{p}')\vec{\sigma}_{\alpha\beta} \cdot \vec{\sigma}_{\alpha'\beta'} \tag{8}$$

a) Expand $\hat{n}_{\vec{p}}$, $\hat{\epsilon}_{\vec{p}}$, and $\hat{f}_{\vec{\sigma}\vec{\sigma}'}(\vec{p},\vec{p}')$ in terms of the unit matrix $\sigma^0 = \mathbf{1}$ and the Pauli spin matrices $\sigma^1 = \sigma^x$, $\sigma^2 = \sigma^y$, $\sigma^3 = \sigma^z$ and find Landau's energy functional E.

b) Assume that the electric field is directed along the z direction. Show that the polarization of such a system is given by

$$P_z = \frac{\partial E}{\partial E_z} = \frac{\mu}{m^* c} \sum_{\vec{p}} \left(p_y \delta n^1_{\vec{p}} - p_x \delta n^2_{\vec{p}} \right). \tag{9}$$

Here, $\delta n^i_{\vec{p}} = \frac{1}{2}\mathrm{tr}\left[\delta\hat{n}_{\vec{p}}\sigma^i\right]$ and $\delta\hat{n}_{\vec{p}}$ is the deviation from the equilibrium ($E_z = 0$) distribution function.

c) The application of the electric field changes the quasiparticle energy in linear response according to

$$\delta\tilde{\epsilon}^i_{\vec{p}} = \delta\epsilon^i_{\vec{p}} + \frac{2}{V}\sum_{\vec{p}'} f^{ii}(\vec{p},\vec{p}')\delta n^i_{\vec{p}'} \quad \text{with} \quad \delta n^i_{\vec{p}} = \frac{\partial n_0}{\partial \epsilon}\delta\tilde{\epsilon}^i_{\vec{p}} = -\delta(\epsilon^0_{\vec{p}} - \epsilon_F)\delta\tilde{\epsilon}^i_{\vec{p}}. \tag{10}$$

Use the ansatz $\delta\tilde{\epsilon}^i_{\vec{p}} = \alpha\delta\epsilon^i_{\vec{p}}$ and show that $\alpha = 1/(1 + F^a_1/3)$ to find $\delta n^i_{\vec{p}}$ and $\delta\tilde{\epsilon}^i_{\vec{p}}$.

d) Compute χ according to Eq. (7).

References

[1] Pomeranchuk, Ia., *Sov. Phys. JETP* **8**, 361 (1958).