Exercise 3.1 Bloch Oscillations

In the quasi-classical description of a wave packet peaked around some quasi-momentum $\hbar k$ the group velocity is given by

$$\dot{r} = \frac{1}{\hbar}\frac{\partial \varepsilon_k}{\partial k} \tag{1}$$

while the change of the quasi-momentum is given by

$$\hbar \dot{k} = F_{\text{ext}} \tag{2}$$

with F_{ext} the force due to applied external fields (in addition to the periodic potential).

a) We focus on the one-dimensional tight-binding model with the dispersion relation

$$\varepsilon_k = 2\mathrm{t}\cos(ka) \tag{3}$$

where t is the nearest neighbor hopping constant and a the lattice constant (for simplicity we consider only one band). Show that the effect of a uniform electric field is, instead of accelerating the electrons, letting them oscillate around some fixed position.
This means that for sufficiently large fields all metals would behave like insulators. Why has this effect never been seen in normal metals? What would change if we considered instead of metals semiconductor superlattices?

b) We now add a small damping term to (2) and analyze the consequences. The rate of change of the quasi-momentum is thus given by

$$\hbar \dot{k} = F_{\text{ext}} - \frac{m\dot{r}}{\tau} \tag{4}$$

where τ is the relaxation time.
Show that this damping can lead to a vanishing of the oscillations and thus to a stationary solution. What is the corresponding condition and how does the stationary solution look like?
Calculate then analytically $k(t)$ for both situations to verify your considerations.

Exercise 3.2 Van-Hove Singularities

We consider an energy band $\epsilon(\mathbf{k})$ in d dimensions. The density of states $\rho(\epsilon)$ is given by

$$\rho(\epsilon)d\epsilon = v_0 \int d^d k \tag{5}$$

and the integration boundaries within one Brillouin zone are defined by $\epsilon \leq \epsilon(\mathbf{k}) \leq \epsilon + d\epsilon$. Show that the density of states can also be calculated with the formula

$$\rho(\epsilon) = v_0 \int_{S(\epsilon)} \frac{dS}{|\nabla\epsilon(\mathbf{k})|} = v_0 \int_{S(\epsilon)} \left[\sum_{\alpha=1}^{d} \left(\frac{\partial\epsilon(\mathbf{k})}{\partial k_\alpha} \right)^2 \right]^{-\frac{1}{2}} dS. \tag{6}$$

$S(\epsilon)$ is respectively the line (for $d = 2$) and the surface (for $d = 3$) in reciprocal space given by the equation $\epsilon(\mathbf{k}) = \epsilon$. dS is the line- and surface-element, respectively.
The singularities in $\rho(\epsilon)$ arise from the critical points of $\epsilon(\mathbf{k})$ where the gradient $\nabla\epsilon(\mathbf{k})$ vanishes.
We now consider $\mathbf{k}_c$, a critical point with a non-vanishing determinant $|\partial^2\epsilon/\partial k_\alpha \partial k_\beta|$.
Applying an appropriate coordinate transformation leaving the volume element $d^d k$ unchanged, the dispersion in the vicinity of $\mathbf{k}_c$ can be written as

$$\epsilon(\mathbf{k}) = \epsilon_c + a \sum_{\alpha=1}^{d} \eta_\alpha \xi_\alpha^2 + \ldots, \qquad \eta_\alpha = \pm 1, \qquad \boldsymbol{\xi} = \mathbf{k} - \mathbf{k}_c, \tag{7}$$

where $a > 0$ and $\epsilon(\mathbf{k}_c) = \epsilon_c$.
Show that for ϵ near ϵ_c we find in two dimensions the singularities

$$\eta_1 = \eta_2 = -1 \qquad \rho(\epsilon) = \begin{cases} C + A & \text{für} \quad \epsilon < \epsilon_c \\ C & \text{für} \quad \epsilon > \epsilon_c \end{cases}$$

$$\eta_1 = -\eta_2 = \pm 1 \qquad \rho(\epsilon) = C - \frac{v_0}{a} \log \left| 1 - \frac{\epsilon}{\epsilon_c} \right|$$

$$\eta_1 = \eta_2 = 1 \qquad \rho(\epsilon) = \begin{cases} C & \text{für} \quad \epsilon < \epsilon_c \\ C + A & \text{für} \quad \epsilon > \epsilon_c \end{cases}$$

and in three dimensions the singularities

$$\eta_1 = \eta_2 = \eta_3 = -1 \qquad \rho(\epsilon) = \begin{cases} C + B(\epsilon_c - \epsilon)^{\frac{1}{2}} & \text{für} \quad \epsilon < \epsilon_c \\ C & \text{für} \quad \epsilon > \epsilon_c \end{cases}$$

$$\eta_1 = \eta_2 = -\eta_3 = -1 \qquad \rho(\epsilon) = \begin{cases} C & \text{für} \quad \epsilon < \epsilon_c \\ C - B(\epsilon - \epsilon_c)^{\frac{1}{2}} & \text{für} \quad \epsilon > \epsilon_c \end{cases}$$

$$\eta_1 = \eta_2 = -\eta_3 = 1 \qquad \rho(\epsilon) = \begin{cases} C - B(\epsilon_c - \epsilon)^{\frac{1}{2}} & \text{für} \quad \epsilon < \epsilon_c \\ C & \text{für} \quad \epsilon > \epsilon_c \end{cases}$$

$$\eta_1 = \eta_2 = \eta_3 = 1 \qquad \rho(\epsilon) = \begin{cases} C & \text{für} \quad \epsilon < \epsilon_c \\ C + B(\epsilon - \epsilon_c)^{\frac{1}{2}} & \text{für} \quad \epsilon > \epsilon_c. \end{cases}$$

Here, $A = \pi v_0/a$ and $B = 2\pi v_0/a^{\frac{3}{2}}$. In addition, the integration domain in (6) was restricted to a small vicinity around $\mathbf{k}_c$. (The constant C depends on the choice of this vicinity) Correction terms of order $O(\epsilon - \epsilon_c)$ were omitted in the above formulae.
Are the singularities occurring here integrable?