Solid State Theory
Exercise 10

Transport in metals

Exercise 10.1 The f-sum rule

The f-sum rule is a consequence of the particle conservation in a given system

$$\int_0^\infty d\omega' \sigma_1(\omega') = \frac{\pi e^2 n}{2m}. \tag{1}$$

It is a useful tool to check the consistency of any approximative treatment. In the following we want to derive the f-sum rule for an electronic system with the (one-particle) Hamilton operator

$$\mathcal{H} = \mathcal{H}_0 + U(\mathbf{r},t), \quad \mathcal{H}_0 = -\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{r}), \quad U(\mathbf{r},t) = A_0 e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} + A_0^* e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega t)}. \tag{2}$$

The time-dependent terms induce transitions between the eigenfunctions of $\mathcal{H}_0$.

a) Use Fermi's golden rule to compute the transition rate $W(\mathbf{q},\omega)$ of the many-particle system.

The transitions between the states of $\mathcal{H}_0$ lead to an energy dissipation of the amount $\hbar\omega W(\mathbf{q},\omega)$. The real part of the conductivity, $\sigma_1(\mathbf{q},\omega)$, is related to the energy dissipation by

$$\sigma_1(\mathbf{q},\omega) = \frac{1}{2}\frac{e^2}{q^2}\frac{\hbar\omega W(\mathbf{q},\omega)}{|A_0|^2 V}. \tag{3}$$

b) Compute the double commutator

$$\left[e^{-i\mathbf{q}\cdot\mathbf{r}}, [\mathcal{H}_0, e^{i\mathbf{q}\cdot\mathbf{r}}]\right] \tag{4}$$

and derive the following equation

$$\frac{1}{V}\sum_{\alpha\beta} f(E_\alpha)\left|\langle\psi_\beta|e^{i\mathbf{q}\cdot\mathbf{r}}|\psi_\alpha\rangle\right|^2 (E_\beta - E_\alpha) = \frac{\hbar^2 q^2}{2m}\frac{n}{2} \tag{5}$$

where E_α and ψ_α are the eigenvalues and eigenfunctions of $\mathcal{H}_0$.

c) Combine Eq. (3) and Eq. (5) in order to proof the f-sum rule Eq. (1).

Exercise 10.2 Penetration depth in a superconductor

We consider a superconductor with a normal-conducting component ρ_n and a superconducting component ρ_s, where $\rho = \rho_n + \rho_s$ is the total electron density. The conductivity of the system is given by the conductivity of the two components, $\sigma = \sigma_n + \sigma_s$, with

$$\sigma_n(\omega) = \rho_n \frac{e^2}{m} \frac{\tau}{1 - i\omega\tau} \quad \text{and} \quad \sigma_s(\omega) = i\rho_s \frac{e^2}{m(\omega + i0^+)}. \tag{6}$$

The density of the superconducting component depends on the temperature in the following way (Gorter-Casimir two-fluid model)

$$\rho_s(T) = \rho \left[1 - \left(\frac{T}{T_c} \right)^4 \right]. \tag{7}$$

a) Use the expression for the dielectric constant of a metal $\varepsilon(\omega) = 1 + (4\pi i/\omega)\sigma(\omega)$ in order to compute the penetration depth $\delta(\omega, T)$ in the limit $\omega\tau \ll 1$.

b) Plot the penetration depth $\delta(\omega, T)$ for small ω as a function of the frequency ω and temperature T. Discuss the limits $T \to T_c$ and $T \to 0$.

c) Show that the f-sum rule also holds for a superconductor with conductivity given in Eq. (6).