

Problem 11.1 Entanglement as a resource

Suppose Alice and Bob share a state ρ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$. We wish to show that entanglement does not increase under local operations and classical communication (LOCC). We measure entanglement using

$$E(A : B) := \frac{1}{2} \min_{\rho_{ABR}} I(A : B|R), \quad (1)$$

where the minimum is taken over all extensions ρ_{ABR} of ρ_{AB} such that $\text{tr}_R(\rho_{ABR}) = \rho_{AB}$.

- a) Given a pure state ρ_{AB} , show that $\rho_{ABR} = \rho_{AB} \otimes \rho_R$.
- b) A maximally entangled state on $\mathcal{H}_A \otimes \mathcal{H}_B$ is a pure state ρ_{AB} with $\text{tr}_A(\rho_{AB}) = \mathbb{1}_B/d_B$ and $\text{tr}_B(\rho_{AB}) = \mathbb{1}_A/d_A$, where $d_A = \dim \mathcal{H}_A$. Calculate $E(A : B)$ for the case when \mathcal{H}_A and \mathcal{H}_B are two-dimensional.

A local operation is a CPM and can be decomposed into three parts: (i) introduce of an extension $\rho_{ABR} \mapsto \rho_{AA'BR} = \rho_{ABR} \otimes \sigma_{A'}$, (ii) apply a unitary on \mathcal{H}_A and finally (iii) do a partial trace over some subsystem of \mathcal{H}_A .

- c) Show that extensions (i) and unitaries (ii) do not change mutual information $I(A : B|R)$.
- d) Show that a partial trace (iii) can only reduce mutual information $I(A : B|R)$.

Note that we did not use the minimum in the definition of $E(A : B)$ so far. Classical communication consists of a measurement on Alice's side followed by communication of the measurement result. Mathematically, Alice prepares a state that is classical in A' using local operations:

$$\rho_{AA'BR} = \sum_i p_i |i\rangle\langle i|_{A'} \otimes \rho_{ABR}^i. \quad (2)$$

Classical communication maps this state to

$$\rho_{AA'BR} \mapsto \rho_{AA'BB'R''} = \sum_i p_i |i\rangle\langle i|_{A'} \otimes |i\rangle\langle i|_B \otimes \rho_{ABR''}^i, \quad (3)$$

where R'' is another extension of the system.

- e) Show that $I(A : B)$ can increase under classical communication by giving an example. Hint: Start with an empty system and share a classical random variable.
- f) Show that $I(A : B|R)$ does not increase in your example for an appropriate choice of extension R .
- g) Show that for any R there exists an R' such that $I(A, A' : B|R) \geq I(A, A' : B, B'|R')$.