ETH Eidgenössische Technische Hochschule Zürich
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# Quantum Information Theory
# Problem Set 7

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## Problem 7.1 Singlet State

Consider Alice and Bob sharing a Bell state $|\Psi\rangle$ on $\mathcal{H}_2^{\otimes 2}$, where $\mathcal{H}_2$ is a two-dimensional Hilbert space with basis $\{|0\rangle, |1\rangle\}$:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{1}$$

a) Show that this state is invariant under simultaneous basis changes, namely $U^{\otimes 2}|\Psi\rangle = e^{i\phi}|\Psi\rangle$ for some phase $\phi$ and for an arbitrary unitary $U$.

b) As a corollary, show that if Alice and Bob measure in the same basis, their results will be perfectly anti-correlated.

## Problem 7.2 Tsirelson's Inequality

Tsirelson's inequality (cf. Nielsen/Chuang, Problem 2.3) gives an upper bound on the possible violation of Bell's inequality in Quantum mechanics. Let $Q = \vec{q} \cdot \vec{X}$, $R = \vec{r} \cdot \vec{X}$, $S = \vec{s} \cdot \vec{X}$ and $T = \vec{t} \cdot \vec{X}$ be observables with $|\vec{q}| = |\vec{r}| = |\vec{s}| = \left|\vec{t}\right| = 1$ and the Pauli matrices $\vec{X}$.

a) Show that all two-dimensional observables $Q$ with eigenvalues $\pm 1$ can be written in the form $Q = \vec{q} \cdot \vec{X}$.

b) Show that
$$(Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2 = 4\mathbb{1} + [Q, R] \otimes [S, T]. \tag{2}$$

c) Show that the expectation value can be bounded as $\langle \sqrt{X} \rangle \leq \sqrt{\lambda_{\max}}$, where $\lambda_{\max}$ is the maximum eigenvalue of a positive $X$.

d) Use the above results to prove Tsirelson's inequality:
$$\langle Q \otimes S \rangle + \langle R \otimes S \rangle + \langle R \otimes T \rangle - \langle Q \otimes T \rangle \leq 2\sqrt{2}. \tag{3}$$

e) We can generalize this to higher dimensions: Show that equation (2) holds for any set of observables with eigenvalues $\pm 1$.