Problem 6.1 Fidelity and Uhlmann's Theorem

Given two states ρ and σ on $\mathcal{H}_{\rm A}$ with fixed basis $\{|i\rangle_{\rm A}\}_i$ and a reference Hilbert space $\mathcal{H}_{\rm B}$ with fixed basis $\{|i\rangle_{\rm B}\}_i$, which is a copy of $\mathcal{H}_{\rm A}$, Uhlmann's theorem claims that the fidelity can be written as

$$F(\rho,\sigma) = \max_{|\psi\rangle} |\langle\psi|\phi\rangle| \,, \tag{1}$$

where the maximum is over all purifications $|\psi\rangle$ of ρ and $|\phi\rangle$ of σ on $\mathcal{H}_{\rm A}\otimes\mathcal{H}_{\rm B}$. Let us introduce a state $|\psi\rangle$ as:

$$|\psi\rangle = (\sqrt{\rho}\otimes U_{\rm B})\,|\gamma\rangle, \qquad |\gamma\rangle = \sum_i |i\rangle_{\rm A}\otimes|i\rangle_{\rm B}, \tag{2}$$

where $U_{\rm B}$ is any unitary on $\mathcal{H}_{\rm B}$.

a) Show that $|\psi\rangle$ is a purification of ρ.

b) Argue why every purification of ρ can be written in this form.

c) Use the construction presented in the proof of Uhlmann's theorem to calculate the fidelity between $\sigma' = \mathbb{1}_2/2$ and $\rho' = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ in the 2-dimensional Hilbert space with computational basis.

d) Give an expression for the fidelity between any pure state and the completely mixed state $\mathbb{1}_{\rm n}/n$ in the n-dimensional Hilbert space.

Problem 6.2 The Choi-Jamiolkowski Isomorphism

The Choi-Jamiolkowski Isomorphism can be used to determine whether a given mapping is a CPM. Consider the family of mappings between operators on two-dimensional Hilbert spaces

$$\mathcal{E}_\alpha : \rho \;\mapsto\; (1-\alpha)\,\frac{\mathbb{1}_2}{2} + \alpha\,\Big(\frac{\mathbb{1}_2}{2} + X\rho Z + Z\rho X\Big), \qquad 0\le\alpha\le 1. \tag{3}$$

a) Use the Bloch representation to determine for what range of α these mappings are positive. What happens to the Bloch sphere?

b) Calculate the analogs of these mappings in state space by applying the mappings to the fully entangled state as follows:

$$\sigma_\alpha = (\mathcal{E}_\alpha\otimes\mathcal{I})\big[|\Psi\rangle\langle\Psi|\big], \qquad |\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{4}$$

For what range of α is the mapping a CPM?

*c) Find an operator-sum representation of $\mathcal{E}_\alpha$ for $\alpha = 1/4$.