
Problem 5.1 Quantum operations can only decrease distance

Given a trace-preserving quantum operation $\mathcal{E} : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ and two states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, show that

$$\delta\left(\mathcal{E}(\sigma), \mathcal{E}(\rho)\right) \leq \delta(\sigma, \rho). \tag{1}$$

What physical principle implies that this statement has to hold?

Problem 5.2 Depolarizing channel

We are given two two-dimensional Hilbert spaces $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{B}}$ and a CPM $\mathcal{E}_p : \mathcal{S}(\mathcal{H}_{\mathrm{A}}) \to \mathcal{S}(\mathcal{H}_{\mathrm{B}})$ defined as

$$\mathcal{E}_p : \rho \mapsto \frac{p}{2}\mathbb{1} + (1-p)\rho. \tag{2}$$

a) Find an operator-sum representation for $\mathcal{E}_p$. Note that $\rho \in \mathcal{S}(\mathcal{H}_{\mathrm{A}})$ can be written in the Bloch sphere representation:

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{X}), \quad \vec{r} \in \mathbb{R}^3, \quad \vec{r} \cdot \vec{X} = r_x X + r_y Y + r_z Z, \tag{3}$$

where X, Y and Z are Pauli matrices.

b) What happens to the radius $\vec{r}$ when we apply $\mathcal{E}_p$? What is the physical interpretation of this?

c) A probability distribution $P_A(0) = q$, $P_A(1) = 1 - q$ can be encoded in a quantum state on $\mathcal{H}_{\mathrm{A}}$ as $\rho = q|0\rangle\langle 0|_{\mathrm{A}} + (1-q)|1\rangle\langle 1|_{\mathrm{A}}$. Calculate $\mathcal{E}(\rho)$ and the conditional probabilities $P_{B|A}$ as well as P_B, which are defined accordingly on $\mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}}$.

d) Maximize the mutual information over q to find the classical channel capacity of the depolarizing channel.