Quantum Information Theory
Problem Set 3

Problem 3.1 Quantum-Telepathy Game

Consider the game of n collaborating players $P_1, P_2, \ldots, P_n$ presented in the lecture. After the players had time to decide on a strategy, two players — let us denote them by P_1 and P_2 — are randomly chosen without knowing of each other and sent into separate rooms. The remaining $n-2$ players know who is missing and are allowed to communicate. They broadcast one bit of information (in the form of two coins that are either both heads or both tails) to the chosen players. The two players must then each decide whether to flip their respective coin and the game is won if the two coins end up in different positions.
Classically, this game is won with a probability of at most 75% for large n. Can we use quantum mechanics to improve this result?
The optimal quantum strategy can be sketched as follows: The players decide to prepare a state in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ and distribute it, leaving player i in control of $\mathcal{H}_i$. When the two players are chosen, the remaining players measure their qubit in some basis and count the number of results of some kind. This number mod 2 is then sent to the chosen players. From this bit, they can deduce the basis in which they have to measure in, such that their results are guaranteed to be different.

a) Let $\{\, |0\rangle,\ |1\rangle \,\}$ denote the computational basis of the two-dimensional Hilbert space $\mathcal{H}_i$. To warm up, show that

$$\begin{aligned} &\{\, |+\rangle := 1/\sqrt{2}\,(|0\rangle + |1\rangle) \ ,\ |-\rangle := 1/\sqrt{2}\,(|0\rangle - |1\rangle) \,\} \qquad \text{and} \\ &\{\, |\circlearrowleft\rangle := 1/\sqrt{2}\,(|0\rangle + i|1\rangle) \ ,\ |\circlearrowright\rangle := 1/\sqrt{2}\,(|0\rangle - i|1\rangle) \,\} \end{aligned} \tag{1}$$

are alternative bases of $\mathcal{H}_i$.

b) We introduce two quantum operators $O_\pm^n : \mathcal{S}\big(\bigotimes_{i=1}^n \mathcal{H}_i\big) \rightarrow \mathcal{S}\big(\bigotimes_{i=1}^{n-1} \mathcal{H}_i\big)$, which first measure a $|+\rangle$ or $|-\rangle$ respectively on the n-th (last) qubit and then use a partial trace to get rid of the n-th subsystem. Give an expression for $O_\pm^n$ assuming that both measurement outcomes have equal probability.

c) Let us introduce two entangled states

$$|\Psi_\pm^n\rangle := \frac{1}{\sqrt{2}}\big(|0\rangle^{\otimes n} \pm |1\rangle^{\otimes n}\big) \tag{2}$$

with density operators $\rho_\pm^n = |\Psi_\pm^n\rangle\langle\Psi_\pm^n|$. Show that

$$O_+^n(\rho_+^n) = \rho_+^{n-1}, \quad O_+^n(\rho_-^n) = \rho_-^{n-1}, \quad O_-^n(\rho_+^n) = \rho_-^{n-1}, \quad \text{and} \quad O_-^n(\rho_-^n) = \rho_+^{n-1}. \tag{3}$$

d) Show that for $n = 2$, the states can be written as

$$|\Psi_+^2\rangle = \frac{1}{\sqrt{2}}\left(|\circlearrowright\circlearrowleft\rangle + |\circlearrowleft\circlearrowright\rangle\right) \quad \text{and} \quad |\Psi_-^2\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right). \tag{4}$$

e) Given above results, work out a detailed quantum strategy that always wins this game.

Problem 3.2 Partial Trace

Given a density matrix ρ_{AB} on the bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{\mathrm{A}} = \mathrm{tr}_{\mathrm{B}}\rho_{\mathrm{AB}}$, show that ρ_{A} is a valid density operator by proving the following properties:

a) Hermiticity — $\rho_{\mathrm{A}} = \rho_{\mathrm{A}}^\dagger$.

b) Positivity — $\rho_{\mathrm{A}} \geq 0$.

c) Normalization — $\mathrm{tr}\rho_{\mathrm{A}} = 1$.

d) Calculate the reduced density matrix of system A of the Bell state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right), \quad \text{where} \quad |ab\rangle = |a\rangle_{\mathrm{A}} \otimes |b\rangle_{\mathrm{B}}. \tag{5}$$

What can you say about this reduced state?