# Quantum Information Theory
# Problem Set 1



## Problem 1.1 Trace distance

The trace distance (or $L_1$-distance) between two probability distributions $P_X$ and $Q_X$ over a discrete alphabet $\mathcal{X}$ is defined as

$$\delta(P_X, Q_X) = \frac{1}{2}\sum_{x\in\mathcal{X}} |P_X(x) - Q_X(x)| \,. \tag{1}$$

The trace distance may also be written as

$$\delta(P_X, Q_X) = \max_{\mathcal{S}\subseteq\mathcal{X}} |P_X[\mathcal{S}] - Q_X[\mathcal{S}]| \,, \tag{2}$$

where we maximize over all events $S \subseteq \mathcal{X}$ and the probability of an event is given by $P_X[\mathcal{S}] = \sum_{x\in\mathcal{S}} P_X(x)$.

a) Show that $\delta(\cdot,\cdot)$ is a good measure of distance by proving that $0 \leq \delta(P_X, Q_X) \leq 1$ and the triangle inequality $\delta(P_X, R_X) \leq \delta(P_X, Q_X) + \delta(Q_X, R_X)$ for arbitrary probability distributions $P_X$, $Q_X$ and $R_X$.

b) Show that definitions (2) and (1) are equivalent and use (2) to give a physical interpretation of the trace distance.

## Problem 1.2 Weak Law of Large Numbers

Let $A$ be a positive random variable with expectation value $\langle A\rangle$ and let $P[A \geq \varepsilon]$ denote the probability of an event $\{A \geq \varepsilon\}$.

a) Prove Markov's inequality

$$P[A \geq \varepsilon] \leq \frac{\langle A\rangle}{\varepsilon}. \tag{3}$$

b) Use Markov's inequality to prove the weak law of large numbers for i.i.d. $X_i$:

$$\lim_{n\to\infty} P\left[\left(\frac{1}{n}\sum_i X_i - \mu\right)^2 \geq \varepsilon\right] = 0 \quad \text{for any } \varepsilon > 0,\ \mu = \langle X_i\rangle. \tag{4}$$

## Problem 1.3 Min-Entropy

The classical min-entropy of a probability distribution $P_X$ over $\mathcal{X}$ is defined as

$$H_{\min}(X) = \min_{x\in\mathcal{X}} h_x, \tag{5}$$

where the information content of an event $\{X = x\}$ is given by $h_x = -\log P_X(x)$. The following lemma has been used in the lecture:

**Lemma 1.** *Let $\lambda \geq 0$ and $2^\lambda \in \mathbb{N}$. If $H_{\min}(X) \geq \lambda$ then there exists a probability distribution $P_R$ such that $P_X(x) = \sum_r P_R(r)P_{X|R=r}(x)$, where $P_{X|R=r}(x)$ is flat and has support of size $2^\lambda$.*

a) Show that $H_{\min}(X) \leq \log|\mathcal{X}|$ for any distribution $P_X$ over $\mathcal{X}$.

*b) Prove Lemma 1.