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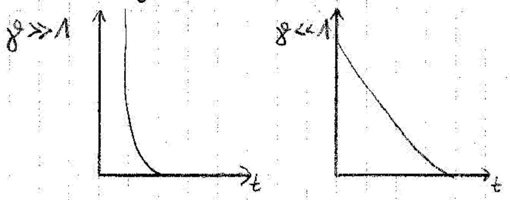
2 Übungsblatt Atom- und Molekülphysik

## 2.1

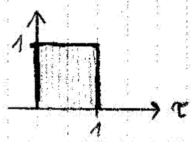
$$\overline{A(t)A(0)} \propto \int_0^\infty A(\tau)A(t+\tau)\,d\tau$$

a) $\int_0^\infty e^{-\gamma\tau} \cdot e^{-\gamma(t+\tau)}\,d\tau = e^{-\gamma t}\int_0^\infty e^{-2\gamma\tau}\,d\tau = e^{-\gamma t}\left[-\frac{1}{2\gamma}e^{-2\gamma\tau}\right]_0^\infty = \frac{1}{2\gamma}e^{-\gamma t}$

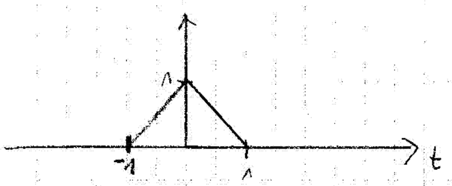



b) We can express the given function with the help of unitstepfunctions.

$\Rightarrow \quad A(\tau) = \Theta(\tau) - \Theta(\tau - 1)$




$$\int_0^\infty (\Theta(\tau) - \Theta(\tau-1))(\Theta(\tau+t) - \Theta(\tau+t-1))\,d\tau = 2t\,\Theta(-t) - (1+t)\Theta(-1-t) - (t-1)\Theta(1-t)$$




The second way is the old fashioned one without heaviside's help:

$$A(\tau) = \begin{cases} 1, & 0 \le \tau \le 1 \\ 0, & \tau > 1 \end{cases}$$

We get a 0 for the integrand, if either $A(\tau)$ or $A(\tau+t)$ equals 0. This means that both of them have to be $\neq 0$. This is only possible for:

$0 \le \tau \le 1$ and $0 \le \tau + t \le 1$ ; for $t > 1$ or $t < -1$ this isn't possible.
$(-t \le \tau \le 1-t)$

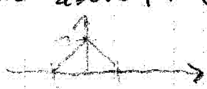
First chase: $0 \le t \le 1$ : $\overline{A(t)A(0)} = \int_0^{1-t} 1\,d\tau = 1-t$,

Second chase: $-1 \le t \le 0$ : $\overline{A(t)A(0)} = \int_{-t}^{1} 1\,d\tau = [\tau]_{-t}^{1} = 1+t$.

That means we receive:

$$A(\tau) = \begin{cases} 1-t, & 0 \le t \le 1 \\ 1+t, & -1 \le t \le 0 \\ 0, & \text{in all other chases} \end{cases}$$

The graph is the same (for sure) like the one above for the treatment of the problem with unit-step-functions.



2.2

Lorentzian: $f(\omega) = \frac{\Gamma_{hom}}{\Gamma_{hom}^2 + (\omega-\omega_0)^2} = \frac{1}{\Gamma_{hom}} \frac{1}{1+\left(\frac{\omega-\omega_0}{\Gamma_{hom}}\right)^2} = \frac{1}{\Gamma_{hom}} \frac{1}{1+x^2}$ ;

with $x^2 = \left(\frac{\omega-\omega_0}{\Gamma_{hom}}\right)^2$.

Gaussian: $g(\omega) = \exp\left(-\ln 2 \cdot \left(\frac{2\omega}{\Gamma_{inh}}\right)^2\right) = e^{-\omega^2 a}$ , with $a = \frac{4\ln 2}{\Gamma_{inh}^2}$.

$F(t)G(t) = \int_{-\infty}^{\infty} e^{-i\omega t} f(\omega-\omega_0) g(\omega) d\omega$ (wrong definition on task paper)

with $F(t) = \int_{-\infty}^{\infty} e^{-i\omega t} f(\omega) d\omega$ and $G(t) = \int_{-\infty}^{\infty} e^{-i\omega t} g(\omega) d\omega$

a) $F(t)G(t)$ is meant to be calculated.

$$F(t) = \int_{-\infty}^{\infty} e^{-i\omega t} \frac{1}{\Gamma_{hom}} \frac{1}{1+\left(\frac{\omega-\omega_0}{\Gamma_{hom}}\right)^2} d\omega \underset{\substack{\uparrow \\ x = \frac{\omega-\omega_0}{\Gamma_{hom}} \Leftrightarrow dx\,\Gamma_{hom} = d\omega}}{=} \int_{-\infty}^{\infty} e^{-i(\Gamma_{hom} x + \omega_0)t} \frac{1}{1+x^2} dx$$

$$\underset{\substack{\uparrow \\ e^{-ix} = (\cos x - i \cdot \sin x)}}{=} e^{-i\omega_0 t} \underbrace{\int_{-\infty}^{\infty} \frac{\cos(\Gamma_{hom} t x)}{1+x^2}}_{\substack{\text{Bronstein (21.15 with } a = \Gamma_{hom} t), \\ \text{factor 2 for symmetric} \\ \text{function from 0 to infinity}}} - \underbrace{i e^{-i\omega_0 t} \int_{-\infty}^{\infty} \frac{\sin(\Gamma_{hom} t x)}{1+x^2}}_{\substack{\text{symmetric borders},\ \text{antisymmetric function} \\ =0}}$$

$$= 2 e^{-i\omega_0 t} \frac{\pi}{2} e^{-|\Gamma_{hom} t|} = \pi \underbrace{e^{-i\omega_0 t}}_{\text{oscillation}} \underbrace{e^{-|\Gamma_{hom} t|}}_{\text{damping}}.$$

$$G(t) = \int_{-\infty}^{\infty} e^{-i\omega t} e^{-\ln 2 \left(\frac{2\omega}{\Gamma_{inh}}\right)^2} d\omega = \underbrace{2\int_0^{\infty} \cos(\omega t)\, e^{-\omega^2 a} d\omega}_{\text{Bronstein (21.27)}} - \underbrace{i \cdot \int_{-\infty}^{\infty} \sin(\omega t)\, e^{-\omega^2 a}}_{\substack{=0 \\ \text{(antisymmetric)}}} \quad a = \frac{4\ln 2}{\Gamma_{inh}^2}$$

$$= \frac{\sqrt{\pi}\ e^{-\frac{\Gamma_{inh}^2 t^2}{16\ln 2}}}{4\sqrt{\ln 2}} \cdot \Gamma_{inh}$$

$$\Rightarrow F(t)G(t) = \frac{\pi^{\frac{3}{2}}}{4\sqrt{\ln 2}} e^{-i\omega_0 t - |\Gamma_{hom} t| - \frac{\Gamma_{inh}^2 t^2}{16 \ln 2}} \cdot \Gamma_{inh}.$$

b)

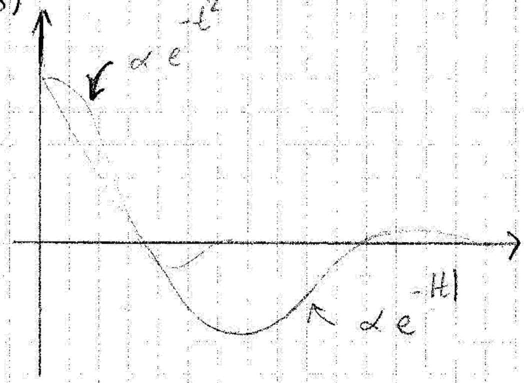
I) $\Gamma_{hom} \gg \Gamma_{inh}$ : $\Rightarrow$ $F(t)G(t) \propto e^{-|t|}$

II) $\Gamma_{hom} \ll \Gamma_{inh}$ : $\Rightarrow$ $F(t)G(t) \propto e^{-t^2}$

We get a damped oscillation in both cases, while the speed till it is damped differs. The inhomogeneous damping is of higher order than the homogeneous damping.

## 2.2

b)

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In[13]:= DisplayTogether[Plot[Cos[16 t] e^-t , {t, 0, 2}], Plot[e^-t , {t, 0, 2}], ImageSize → 550]
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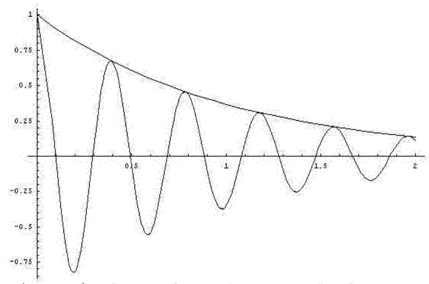


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In[16]:= DisplayTogether[Plot[Cos[16 t] Exp[-t^2], {t, 0, 2}], Plot[Exp[-t^2], {t, 0, 2}], ImageSize → 550]
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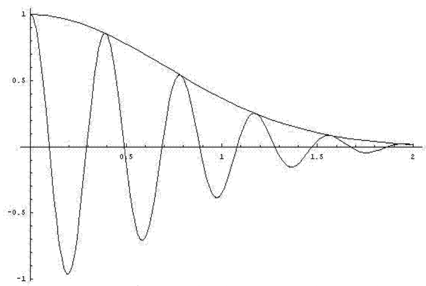


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DisplayTogether[Plot[Cos[t] e^-Abs[t] , {t, 0, 5}, ImageSize → 550], Plot[Cos[t] Exp[-t^2], {t, 0, 5}, ImageSize → 550]]
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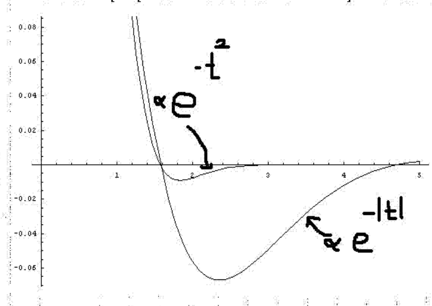

## 2.3

a) normalised lorentzian (1.160): $g(\nu)_{LN} = \dfrac{(2/\pi\Delta\nu)}{1+[2(\nu-\nu_0)/\Delta\nu]^2}$.

This can be rewritten into: $g(\omega) = \frac{2}{\pi}\frac{a}{a^2+\omega^2}$ using $\Delta\nu = a$ and $2(\nu-\nu_0) = \omega$. This one is well known from the script (1.136), only a factor differs for the FT, which is given with $g(t) = \sqrt{\frac{2}{\pi}}e^{-at}$.

Using the property of convolutions from task 2, we can write for the convolution of two normalized lorentzians:

$$S(t) = G^2(t) = \int_{-\infty}^{\infty} e^{-i\omega t} g(\omega-\omega_0)g(\omega)\,d\omega \,, \text{ with}$$

$$G(t) = \int_{-\infty}^{\infty} e^{-i\omega t} g(\omega)\,d\omega \,, \text{ this means we receive}$$

$$\Rightarrow S(t) = \left(\int_{-\infty}^{\infty} e^{-i\omega t} g(\omega)\,d\omega\right)^2 \underset{(1.136)}{\overset{\uparrow}{=}} \left(\sqrt{\frac{2}{\pi}}\,e^{-at}\right)^2 = \frac{2}{\pi}e^{-2at}.$$

We need $S(\omega)$ which is the FT of $S(t)$, with:

$$S(\omega) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\frac{2}{\pi}e^{-2at}}_{(1.136) \text{ with } a\to 2a \text{ and a factor}} = \frac{1}{\pi^2}\left(\sqrt{\frac{\pi}{2}}\,\frac{2a}{4a^2+\omega^2}\right).$$

The maximum can be found at $\omega = 0$, which is $S(\omega)|_{max} = \sqrt{\frac{\pi}{2}}\frac{1}{2\pi^2 a}$. This leads to a half maximum of $\sqrt{\frac{\pi}{2}}\frac{1}{4\pi^2 a}$. So we get a FWHM of:

$$\sqrt{\frac{\pi}{2}}\frac{1}{4\pi^2 a} = \frac{1}{\pi^2}\sqrt{\frac{\pi}{2}}\frac{2a}{4a^2+\omega^2}$$

$$\Leftrightarrow \quad 4a^2+\omega^2 = 8a^2$$

$$\Leftrightarrow \quad \omega = \pm 2a \,,$$

which means we get $\Delta\omega = 4a$. For the normalized lorentzian without the convolution we get a max at $\omega = 0$ with $g(\omega)|_{max} = \frac{2}{\pi a}$, leading to half max of $\frac{1}{\pi a}$. So the FHWM is

$$\frac{1}{\pi a} = \frac{2}{\pi}\frac{a}{a^2+\omega^2}$$

$$\Leftrightarrow \quad a^2+\omega^2 = 2a^2$$

$$\Leftrightarrow \quad \omega = \pm a$$

which means we get $\Delta\omega = 2a$, which is the half of what we got for the convolution.

b) With the knowledge of part a) we already know, that the convolution of the lorentzians gives us a doubled FWHM. While holeburning we produce this lorentzian shaped curve by „destroying" some of the ~~atoms~~, which are ~~able~~ resonant to the $\lambda$ of the laser. Afterwards they aren't there to absorb $\lambda$'s anymore and that's why we get less absorbtion.

2.4

to show: $\langle \delta A(t)\, \delta A(0) \rangle_c = \langle A(t)\, A(0) \rangle_c - \langle A \rangle_c^2$

with $\delta A(t) = A(t) - \langle A(t) \rangle_c$ and $\langle A(t) \rangle_c = \langle A(0) \rangle_c = \langle A \rangle_c$

$\Rightarrow$

$$\langle \delta A(t) \delta A(0) \rangle_c = \langle (A(t) - \langle A(t) \rangle_c)(A(0) - \langle A(0) \rangle_c) \rangle_c$$

$$= \langle A(t) A(0) - A(t) \langle A \rangle_c - A(0) \langle A \rangle_c + \langle A \rangle_c^2 \rangle_c$$

$$= \langle A(t) A(0) \rangle_c - \underbrace{\langle A(t) \rangle_c}_{\langle A \rangle_c} \langle A \rangle_c \underbrace{- \underbrace{\langle A(0) \rangle_c}_{\langle A \rangle_c} \langle A \rangle_c + \langle A \rangle_c^2}_{=0}$$

$$= \langle A(t) A(0) \rangle_c - \langle A \rangle_c^2 \quad \square$$